UNIVERSITAT POLITÈCNICA DE CATALUNYA BARCELONATECH

# Computational Structural Mechanics and Dynamics Assignment 1

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Master's in Numerical Methods in Engineering

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February $17^{th}$, 2020

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# 1 Problem Description

## 1.1 Assignment 1.1

Consider the truss problem defined in Figure 1. All geometric and material properties L, $\alpha$, E and A, as well as the applied force P and H, are to be kept as variables. This truss has 8 degrees of freedom (DoF's), with six of them removable by the fixed-displacement conditions at nodes 2, 3 and 4. This structure is statically indeterminate as long as $\alpha \neq 0$.

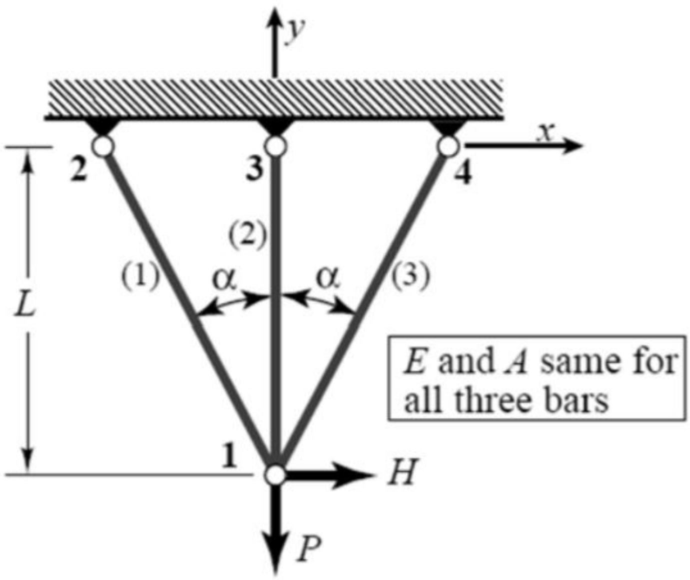


Figure 1: Truss Structure. Geometry and Mechanical Features

$$\frac{EA}{L}\begin{bmatrix} 2cs^2 & 0 & -cs^2 & c^2s & 0 & 0 & -cs^2 & -c^2s \\ & 1+2c^3 & c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \\ & & cs^2 & -c^2s & 0 & 0 & 0 & 0 \\ & & & -c^3 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 \\ & & & & & & cs^2 & c^2s \\ & & & & & & & c^3 \end{bmatrix}\begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix} = \begin{bmatrix} H \\ -P \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

in which c = cos$\alpha$ and s = sin$\alpha$. Explain from physics why the 5th row and column contain only zeros.

(b) Apply the BCs and show the 2-equation modified stiffness system.

(c) Solve for the displacements $u_{x1}$ and $u_{y1}$. Check that the solution makes physical sense for the limit cases $\alpha \to 0$ and $\alpha \to \pi/2$. Why does $u_{x1}$ "blow up" if H $\neq$ 0 and $\alpha \to 0$?

(d) Recover the axial forces in the three members. Partial answer: $F^{(3)}$ = -H/(2s)+P$c^2$/(1+2$c^3$). Why do $F^{(1)}$ and $F^{(3)}$ "blow up" if H $\neq$ 0 and $\alpha \to 0$?

### 1.2 Assignment 1.2

Dr. Who proposes "improving" the result for the example truss of the $1^{st}$ lesson by putting one extra node, 4 at the midpoint of member (3) 1-3, so that it is subdivided in two different members: (3) 1-4 and (4) 3-4. His "reasoning" is that more is better. Try Dr. Who's suggestion by hand computations and verify that the solution "blows up" because the modified master stiffness is singular. Explain physically.

## 2 Solution

### 2.1 Assignment 1.1

#### 2.1.1 Nodal Coordinates and Connectivity

In order to compute the stiffness equations of the structural system, we must first obtain the node coordinates and connectivity from the structural configuration. This will also allow us to compute the length and orientation of each structural element.

| Node | x | y |
|---|---|---|
| 1 | 0 | -L |
| 2 | -Lsin($\alpha$) | 0 |
| 3 | 0 | 0 |
| 4 | Lsin($\alpha$) | 0 |

Table 1: Node Coordinates

| Element | Node 1 | Node 2 | Length | Orientation (rad) |
|---|---|---|---|---|
| 1 | 1 | 2 | L/cos($\alpha$) | $\frac{\pi}{2}+\alpha$ |
| 2 | 1 | 3 | L | $\frac{\pi}{2}$ |
| 3 | 1 | 4 | L/cos($\alpha$) | $\frac{\pi}{2}-\alpha$ |

Table 2: Node Connectivity and Element Geometry

Now we have sufficient information to compute the master stiffness equations of the system.

#### 2.1.2 Stiffness Matrix Computation

The stiffness matrix for any given element in global coordinates in a 2D truss structure is given by:

$$k^{(e)} = \frac{EA}{l^{(e)}} \begin{bmatrix} \eta^2 & \eta\mu & -\eta^2 & -\eta\mu \\ & \mu^2 & -\eta\mu & -\mu^2 \\ & & \eta^2 & \eta\mu \\ & & & \mu^2 \end{bmatrix} \tag{2}$$

Where:

$l^{(e)}$ is the length of element e.

$\eta = cos(\theta)$

$\mu = sin(\theta)$

$\theta$ is the angle of orientation of the element with respect to the positive global x axis.

This way, we may compute the stiffness matrix ($k^{(e)}$) for each element and its contribution to the global stiffness matrix of the structural system ($\Delta K^{(e)}$). In order to simplify the elemental matrices, we will also consider the following trigonometrical identities:

$$cos(\frac{\pi}{2} - \alpha) = sin(\alpha) = s \tag{3}$$

$$sin(\frac{\pi}{2} - \alpha) = cos(\alpha) = c \tag{4}$$

$$sin(\frac{\pi}{2} - \alpha) = sin(\frac{\pi}{2} + \alpha) \tag{5}$$

$$cos(\frac{\pi}{2} - \alpha) = -cos(\frac{\pi}{2} + \alpha) \tag{6}$$

- Element 1

$$k^{(1)} = \frac{EA}{L}c \begin{bmatrix} s^2 & -cs & -s^2 & cs \\ & c^2 & cs & -c^2 \\ & & s^2 & -cs \\ & & & c^2 \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} cs^2 & -c^2s & -cs^2 & c^2s \\ & c^3 & c^2s & -c^3 \\ & & cs^2 & -c^2s \\ & & & c^3 \end{bmatrix} \tag{7}$$

Since Element 1 is between nodes 1 and 2, its corresponding DoFs in the global system are 1, 2,3 and 4. Hence, ($\Delta K^{(1)}$) becomes:

$$\Delta K^{(1)} = \frac{EA}{L} \begin{bmatrix} cs^2 & -c^2s & -cs^2 & c^2s & 0 & 0 & 0 & 0 \\ & c^3 & c^2s & -c^3 & 0 & 0 & 0 & 0 \\ & & cs^2 & -c^2s & 0 & 0 & 0 & 0 \\ & & & c^3 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 0 & 0 & 0 \\ & & & & & & 0 & 0 \\ & & & & & & & 0 \end{bmatrix} \tag{8}$$

- Element 2

$$k^{(2)} = \frac{EA}{L} \begin{bmatrix} 0 & 0 & 0 & 0 \\ & 1 & 0 & -1 \\ & & 0 & 0 \\ & & & 1 \end{bmatrix} \tag{9}$$

$$\Delta K^{(2)} = \frac{EA}{L} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ & & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 \\ & & & & & & 0 & 0 \\ & & & & & & & 0 \end{bmatrix} \tag{10}$$

- Element 3

$$k^{(3)} = \frac{EA}{L}c\begin{bmatrix} s^2 & cs & -s^2 & -cs \\ & c^2 & -cs & -c^2 \\ & & s^2 & cs \\ & & & c^2 \end{bmatrix} = \frac{EA}{L}\begin{bmatrix} cs^2 & c^2s & -cs^2 & -c^2s \\ & c^3 & -c^2s & -c^3 \\ & & cs^2 & c^2s \\ & & & c^3 \end{bmatrix} \tag{11}$$

$$\Delta K^{(3)} = \frac{EA}{L}\begin{bmatrix} cs^2 & c^2s & 0 & 0 & 0 & 0 & -cs^2 & -c^2s \\ & c^3 & 0 & 0 & 0 & 0 & -c^2s & -c^3 \\ & & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 0 & 0 & 0 \\ & & & & & & cs^2 & c^2s \\ & & & & & & & c^3 \end{bmatrix} \tag{12}$$

After all the elemental matrices are known, the global stiffness matrix may be computed as:

$$K = \sum_{e=1}^{n} \Delta K^{(e)} \tag{13}$$

$$\boxed{K = \frac{EA}{L}\begin{bmatrix} 2cs^2 & 0 & -cs^2 & c^2s & 0 & 0 & -cs^2 & -c^2s \\ & 1+2c^3 & c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \\ & & cs^2 & -c^2s & 0 & 0 & 0 & 0 \\ & & & c^3 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 \\ & & & & & & cs^2 & c^2s \\ & & & & & & & c^3 \end{bmatrix}} \tag{14}$$

It may be observed that the fifth row and column of the stiffness matrix are zeros. This may be explained by the fact that they correspond to the horizontal degree of freedom in node 3, which is only connected to element 2. Since element 2 is completely vertical, and we are dealing with a 2D truss structure, this element has zero stiffness in the x direction, therefore making the global stiffness matrix take this form.

### 2.1.3 Displacements and Reaction Forces

Since K is a singular matrix (det(K)=0), the system of equations cannot be solved as it is. By applying boundary conditions, we may reduce the system of equations to one that can be solved by inverting the reduced K in the following way:

$$\left[\begin{array}{c|c} K_{aa} & K_{ab} \\ \hline K_{ba} & K_{bb} \end{array}\right]\begin{bmatrix} U_a \\ U_b \end{bmatrix} = \begin{bmatrix} F_a \\ F_b \end{bmatrix} \tag{15}$$

Where:

$a = [3\ \ 4\ \ 5\ \ 6\ \ 7\ \ 8]$ are the degrees of freedom with known displacements.

$b = [1\ \ 2]$ are the degrees of freedom with unknown displacements.

The unknowns of the problem become $U_b$ (displacements of non-restrained nodes), and $F_a$ (reaction forces). After rearranging the system in this way, the displacements and reaction forces are given by:

$$U_b = K_{bb}^{-1}(F_b - K_{ba}U_a) \tag{16}$$

$$F_a = K_{aa}U_a + K_{ab}U_b \tag{17}$$

For this particular problem, the components of the 2-equation modified stiffness system given by equation (16) become:

$$U_b = \begin{bmatrix} u_{x1} \\ u_{y1} \end{bmatrix} \tag{18}$$

$$U_a = \begin{bmatrix} u_{x2} & u_{y2} & u_{x3} & u_{y3} & u_{x4} & u_{y4} \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \tag{19}$$

$$K_{bb} = \frac{EA}{L}\begin{bmatrix} 2cs^2 & 0 \\ 0 & 1+2c^3 \end{bmatrix} \tag{20}$$

$$K_{ba} = \begin{bmatrix} -cs^2 & c^2s & 0 & 0 & -cs^2 & -c^2s \\ c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \end{bmatrix} \tag{21}$$

$$F_b = \begin{bmatrix} H \\ -P \end{bmatrix} \tag{22}$$

Hence, the displacements take the form:

$$\boxed{U_b = \begin{bmatrix} u_{x1} \\ u_{y1} \end{bmatrix} = \frac{L}{EA}\begin{bmatrix} H/(2cs^2) \\ -P/(2c^3+1) \end{bmatrix}} \tag{23}$$

Now, we will analyze how the solution is affected in the limit cases $\alpha \to 0$ and $\alpha \to \pi/2$.

- Limit case $\alpha \to 0$

  When $\alpha$ goes to zero, $\sin(\alpha)$ goes to zero as well. Therefore, $u_{x1}$ goes to infinity for any H$\neq$0. This is a consequence of the fact that in a 2D truss structure model, elements only have axial stiffness. When $\alpha$=0, all structural elements are completely vertical, which results in a structure with zero stiffness in the x direction that behaves like a pendulum.

- Limit case $\alpha \to \pi/2$

  When $\alpha$ goes to $\pi/2$, $\cos(\alpha)$ goes to zero and $u_{x1}$ goes to infinity like in the previous case. Unlike in the previous case, the structure has elements that should provide stiffness in the x direction: elements 1 and 3 are completely horizontal when $\alpha = \pi/2$. However, since the length of these elements is given by L/$\cos(\alpha)$, it goes to infinity and therefore their stiffness goes to zero, resulting in a pendulum behavior like in the previous case. The difference between the two limit scenarios shows in $u_{y1}$, which is 3 times higher when $\alpha = \pi/2$.

After finding the displacements, we may obtain the reaction forces by applying equation (17):

$$F_a = K_{aa}U_a + K_{ab}U_b = \begin{bmatrix} f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \\ f_{x4} \\ f_{y4} \end{bmatrix} = \begin{bmatrix} \frac{-H}{2} - \frac{Pc^2s}{2c^3+1} \\ \frac{Pc^3}{2c^3+1} + \frac{Hc}{2s} \\ 0 \\ \frac{-P}{2c^3+1} \\ \frac{-H}{2} + \frac{Pc^2s}{2c^3+1} \\ \frac{Pc^3}{2c^3+1} - \frac{Hc}{2s} \end{bmatrix} \tag{24}$$

Then, after computing the reaction forces in nodes 2, 3 and 4, we may calculate the axial force for all structural members.

$$F^{(1)} = -f_{x2}s + f_{y2}c = \frac{H}{2s} + \frac{Pc^2}{2c^3+1} \tag{25}$$

$$F^{(2)} = f_{y3} = \frac{-P}{2c^3+1} \tag{26}$$

$$F^{(3)} = f_{x4}s + f_{y4}c = \frac{-H}{2s} + \frac{Pc^2}{2c^3+1} \tag{27}$$

It may be observed that when $\alpha \to 0$, $F^{(1)}$ and $F^{(3)}$ "blow up". Just like it was discussed for the displacements, this is due to the fact that the structure behaves like a pendulum and cannot resist any horizontal force H≠0. Since the reaction force in the y direction depends on s, it becomes indeterminate.

## 2.2 Assignment 1.2

The structural system proposed by Dr. Who has the following configuration:

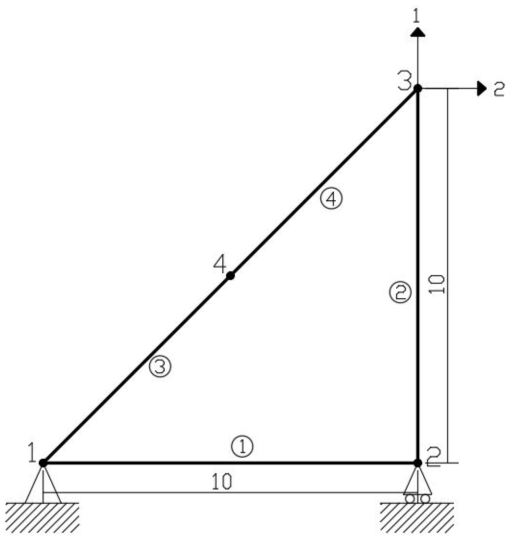


Figure 2: Triangular Truss. Structural Configuration

In this case, elements do not share a common cross section. The geometrical and material properties of each element are illustrated on the following table:

| Element | EA |
|---|---|
| 1 | 100 |
| 2 | 50 |
| 3 | $200\sqrt{2}$ |
| 4 | $200\sqrt{2}$ |

Table 3: Triangular Truss. Material and Geometrical Properties

### 2.2.1 Nodal Coordinates and Connectivity

For this particular structure, the nodal coordinates and connectivity are:

| Node | x | y |
|---|---|---|
| 1 | 0 | 0 |
| 2 | 10 | 0 |
| 3 | 10 | 10 |
| 4 | 4 | 4 |

Table 4: Triangular Truss. Node Coordinates

| Element | Node 1 | Node 2 | Length | Orientation (rad) |
|---|---|---|---|---|
| 1 | 1 | 2 | 10 | 0 |
| 2 | 2 | 3 | 10 | $\frac{\pi}{2}$ |
| 3 | 1 | 4 | $5\sqrt{2}$ | $\frac{\pi}{4}$ |
| 4 | 3 | 4 | $5\sqrt{2}$ | $-\frac{3\pi}{4}$ |

Table 5: Triangular Truss. Node Connectivity and Element Geometry

### 2.2.2 Stiffness Matrix Computation

Following the same process as in Assignment 1.1, we compute the global stiffness matrix of the structural system and reduce it to find the displacements.

The resulting global stiffness matrix is:

$$K = \begin{bmatrix} 50 & 40 & -10 & 0 & 0 & 0 & -40 & -40 \\ & 40 & 0 & 0 & 0 & 0 & -40 & -40 \\ & & 10 & 0 & 0 & 0 & 0 & 0 \\ & & & 5 & 0 & -5 & 0 & 0 \\ & & & & 20 & 20 & -20 & -20 \\ & & & & & 25 & -20 & -20 \\ & & & & & & 60 & 60 \\ & & & & & & & 60 \end{bmatrix} \tag{28}$$

In order to reduce the system, we must apply the boundary conditions:

$a = [1 \;\; 2 \;\; 4]$ (DoFs with known displacements)

$b = [3\ \ 5\ \ 6\ \ 7\ \ 8]$ (DoFs with unknown displacements)

$F_b = [0\ \ 2\ \ 1\ \ 0\ \ 0]^T$ (Applied forces)

Therefore, the reduced matrix $K_{bb}$ takes the form:

$$K_{bb} = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 \\ & 20 & 20 & -20 & -20 \\ & & 25 & -20 & -20 \\ & & & 60 & 60 \\ & & & & 60 \end{bmatrix} \tag{29}$$

However, it may be observed that the matrix is singular (det($K_{bb}$)=0) and therefore, it cannot be inverted and system cannot be solved. This is a consequence of the fact that node 4 is only connected to two elements that have exactly the same orientation and as a result, any force in the direction perpendicular to that of the elements, will cause the solution to blow up. Node 4 does not have any stiffness in the direction perpendicular to elements 3 and 4.

# 3 Discussion

- The most obvious conclusion we may draw out Assignment 1.2 is that more is not always better. Adding the extra node completely changes the behavior of the element it is dividing and makes the system unsolvable. To put it in simple words, this change keeps the model from simulating the behavior of the real structure it is intended to in an accurate way and hence, it is not useful.
- Computation mistakes aside, mathematical errors or indeterminations in a system have a physical cause, which reflects in an abnormal behavior of the analyzed structure. The engineer performing the analysis should be able to identify and make an accurate interpretation of these errors and overall, to go beyond the application of a method and truly analyze the results.

# 4 Appendix 1: Classwork

This section contains the solution to the classwork assigned during the lecture of the $10^{th}$ of February, which consists of solving the following structure:

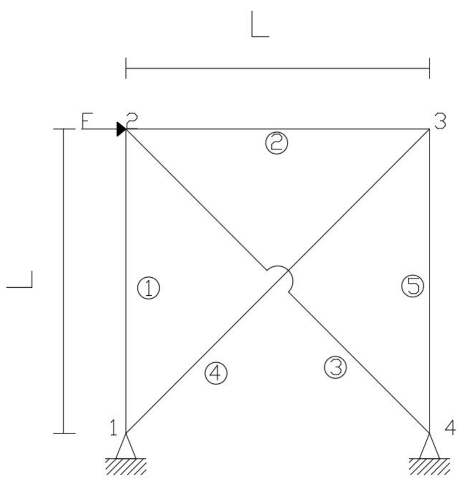


Figure 3: Classwork 2D Frame Structure

All elements have the same material and cross section, with E = 200 GPa and A = 6 $cm^2$. The dimension L is 6 m and force F is 80 KN. Note that the diagonal elements do not intersect one another in the center. Following the same process as in Assignment 1.1, we build the node coordinates and connectivity matrices to then compute the stiffness matrix of the structure.

| Node | x (mm) | y(mm) |
|---|---|---|
| 1 | 0 | 0 |
| 2 | 0 | 6000 |
| 3 | 6000 | 6000 |
| 4 | 6000 | 0 |

Table 6: Node Coordinates, Classwork Frame Structure

| Element | Node 1 | Node 2 | Length (mm) | Orientation (rad) |
|---|---|---|---|---|
| 1 | 1 | 2 | 6000 | $\frac{\pi}{2}$ |
| 2 | 2 | 3 | 6000 | 0 |
| 3 | 2 | 4 | 8485.3 | $-\frac{\pi}{4}$ |
| 4 | 1 | 3 | 8485.3 | $\frac{\pi}{4}$ |
| 5 | 3 | 4 | 6000 | $-\frac{\pi}{2}$ |

Table 7: Node Connectivity and Element Geometry, Classwork Frame Structure

With this information, and applying equations (2) and (13), we compute the global stiffness matrix of the structural system:

$$K = \frac{\sqrt{2}}{2}10^4 \begin{bmatrix} 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 \\ & 3.8284 & 0 & -2.8284 & -1 & -1 & 0 & 0 \\ & & 3.8284 & -1 & -2.8284 & 0 & -1 & 1 \\ & & & 3.8284 & 0 & 0 & 1 & -1 \\ & & & & 3.8284 & 1 & 0 & 0 \\ & & & & & 3.8284 & 0 & -2.8284 \\ & & & & & & 1 & -1 \\ & & & & & & & 3.8284 \end{bmatrix} N/mm \tag{30}$$

Then, we reduce the system by applying boundary conditions.

$a = [1\ \ 2\ \ 7\ \ 8]$ (DoFs with known displacements)

$b = [3\ \ 4\ \ 5\ \ 6]$ (DoFs with unknown displacements)

$F_b = [80000\ \ 0\ \ 0\ \ 0]^T$ (Applied forces in N)

Therefore, the reduced matrix $K_{bb}$ takes the form:

$$K_{bb} = \frac{\sqrt{2}}{2}10^4 \begin{bmatrix} 3.8284 & -1 & -2.8284 & 0 \\ & 3.8284 & 0 & 0 \\ & & 3.8284 & 1 \\ & & & 3.8284 \end{bmatrix} N/mm \tag{31}$$

Then, applying equation (16) we compute the displacements.

$$\boxed{U_b = \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x4} \\ u_{y4} \end{bmatrix} = \begin{bmatrix} 8.5413 \\ 2.2310 \\ 6.7724 \\ -1.7690 \end{bmatrix} mm} \tag{32}$$

# 5 References


- Hurtado Gómez, J.E. Análisis Matricial de Estructuras. Universidad Nacional de Colombia.