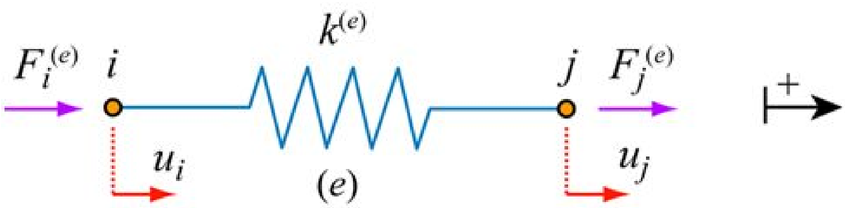
Assignment 10

Computational Structural Mechanics and Dynamics

Sebastian Ares de Parga Regalado

Master in Numerical Methods in Engineering
Universitat Politècnica de Catalunya
April 2020

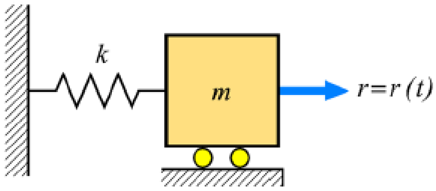


Contents

On "Solid and Structural Dynamics":

1 Assignment 10.1

In the dynamic system of slide 6, let $r(t)$ be a constant force F. What is the effect of F on the time-dependent displacement $u(t)$ and the natural frequency of vibration of the system?



Form Newton's second law:

$$ku + m\ddot{u} = F$$

Considering the free (F=0) undamped vibration:

$$ku + m\ddot{u} = 0 \tag{1}$$

The solution of our problem will be:

$$u = \bar{u}sin(wt)$$

Where:

- $\bar{u}$: is the amplitude of motion.
- w : is the natural frequency of vibration.

Therefore:

$$\ddot{u} = -\bar{u}w^2 sin(wt)$$

Substituting into 1, with F=0:

$$k(\bar{u}sin(wt)) - m(\bar{u}w^2 sin(wt)) = 0$$

Therefore:

$$w = \sqrt{\frac{k}{m}}$$

Now for the Undamped, Forced Vibrations:

$$ku + m\ddot{u} = F \tag{2}$$

This is just a nonhomogeneous differential equation and we know how to solve these. The general solution will be:

$$u(t) = u_c(t) + U_P(t)$$

where the complementary solution is the solution to the free, undamped vibration case. To get the particular solution we can use either undetermined coefficients or variation of parameters depending on which we find easier for a given forcing function. For a constant force, the solution yields:

$$u(t) = \frac{F - Fcos(wt)}{k}$$

$$\ddot{u} = \frac{Fw^2cos(wt)}{k}$$

Substituting into 2:

$$k(\frac{F - Fcos(wt)}{k}) + m(\frac{Fw^2cos(wt)}{k}) = F$$

Therefore:

$$w = \frac{\pi(2n-1)}{2t} \quad and \quad k \neq 0 \quad and \quad t \neq 0 \quad n \in \mathbb{Z}$$

$$w = \pm\sqrt{\frac{k}{m}} \quad and \quad \sqrt{m} \neq 0 \quad and \quad k \neq 0$$

There is a particular type of forcing function that we should take a look at since it leads to some interesting results. Let's suppose that the forcing function is a simple periodic function of the form:

$$F(t) = F_0cos(wt)$$

Therefore:

$$ku + m\ddot{u} = F_0cos(wt)$$

The complementary solution will be:

$$u_c(t) = c_1cos(w_0t) + c_2sin(w_0t)$$

For the particular solution, we will have to check for the cases:

- $w_0 \neq w$
- $w_0 = w$

For the first case our displacement function consists of two cosines and is nice and well behaved for all time.
In contrast, the second case, will have some serious issues at t increases. The addition of the t in the particular solution will mean that we are going to see an oscillation that grows in amplitude as t increases. This case is called resonance and we would generally like to avoid this at all costs.

2 Assignment 10.2

A weight whose mass is m is placed at the middle of a uniform axial bar of length L that is clamped at both ends. The mass of the bar may be neglected. Estimate the natural frequency of vibration in terms of m, L, E and A.
Suggestion: First determine the effective k.
The maximum displacement will be presented at $x = \frac{L}{2}$, and is defined by:

$$u_{\frac{L}{2}} = \frac{FL^3}{192EI}$$

If we let the beam be square:

$$I = \frac{bh^3}{12} = \frac{A^2}{12}$$

Substituting into the maximum displacement, it yields:

$$u_{\frac{L}{2}} = \frac{FL^3}{192E\frac{A^2}{12}} = \frac{FL^3}{16EA^2}$$

Therefore, the effective stiffness matrix is:

$$k = \frac{F}{\frac{FL^3}{16EA^2}} = \frac{16EA^2}{L^3}$$

Since we have a constant force applied to the system:

$$w = \sqrt{\frac{16EA^2}{mL^3}} = \frac{4A}{L}\sqrt{\frac{E}{mL}}$$

3 Assignment 10.3

Use the expression on slide 18 to derive the mass matrix of slide 17. Consistent element mass:

$$m = \int N^T N \rho dV$$

For a linear Lagrange (isoparametric) element with two nodes at $\xi = -1$ and $\xi = 1$ we obtain the following shape functions:

$$N_1 = \frac{1}{2}(1-\xi)$$

$$N_2 = \frac{1}{2}(1+\xi)$$

Therefore:

$$m = \int_{-1}^{1} \begin{bmatrix} \frac{1}{2}(1-\xi) \\ \frac{1}{2}(1+\xi) \end{bmatrix} \begin{bmatrix} \frac{1}{2}(1-\xi) & \frac{1}{2}(1+\xi) \end{bmatrix} \rho A |J| d\xi$$

Where the Jacobian is:

$$|J| = \frac{L}{2}$$

Therefore:

$$m = \frac{\rho AL}{8} \int_{-1}^{1} \begin{bmatrix} (1-\xi)^2 & (1-\xi)(1+\xi) \\ (1-\xi)(1+\xi) & (1+\xi)^2 \end{bmatrix} d\xi$$

$$m = \frac{\rho AL}{8} \begin{bmatrix} \frac{(1-\xi)^3}{3} & \xi - \frac{\xi^3}{3} \\ \xi - \frac{\xi^3}{3} & \frac{(1+\xi)^3}{3} \end{bmatrix}_{-1}^{1}$$

$$m = \frac{\rho AL}{8} \begin{bmatrix} \frac{8}{3} & \frac{4}{3} \\ \frac{4}{3} & \frac{8}{3} \end{bmatrix}$$

Therefore, we obtain:

$$m = \begin{bmatrix} \frac{\rho AL}{3} & \frac{\rho AL}{6} \\ \frac{\rho AL}{6} & \frac{\rho AL}{3} \end{bmatrix}$$

4 Assignment 10.4

Obtain also the mass matrix of a two-node, linear displacement element with a variable cross-sectional area that varies from A_1 to A_2. The approximation of the area can be defined as follows:

$$A(\xi) = \sum N_i(\xi) A_i$$

Therefore:

$$m = \int_{-1}^{1} \begin{bmatrix} \frac{1}{2}(1-\xi) \\ \frac{1}{2}(1+\xi) \end{bmatrix} \begin{bmatrix} \frac{1}{2}(1-\xi) & \frac{1}{2}(1+\xi) \end{bmatrix} \rho(\frac{A_1}{2}(1-\xi) + \frac{A_2}{2}(1+\xi))|J|d\xi$$

$$m = \frac{\rho L}{8} \int_{-1}^{1} \begin{bmatrix} (1-\xi)^2 & (1-\xi)(1+\xi) \\ (1-\xi)(1+\xi) & (1+\xi)^2 \end{bmatrix} (\frac{A_1}{2}(1-\xi) + \frac{A_2}{2}(1+\xi))d\xi$$

$$m = \frac{\rho L}{16} \int_{-1}^{1} A_1 \begin{bmatrix} -(\xi-1)^3 & (\xi-1)^2(\xi+1) \\ (\xi+1)(\xi-1)^2 & -(\xi+1)^2(\xi-1) \end{bmatrix} + A_2 \begin{bmatrix} (\xi-1)^2(\xi+1) & -((\xi+1)^2(\xi-1) \\ -(\xi+1)^2(\xi-1) & (\xi+1)^3 \end{bmatrix}$$

$$m = \frac{\rho L}{16}[A_1 \begin{bmatrix} -\frac{\xi^4}{4} + \xi^3 - \frac{3\xi^2}{2} + \xi & \frac{\xi^4}{4} - \frac{\xi^3}{3} - \frac{\xi^2}{2} + \xi \\ \frac{\xi^4}{4} - \frac{\xi^3}{3} - \frac{\xi^2}{2} + \xi & -\frac{\xi^4}{4} - \frac{\xi^3}{3} + \frac{\xi^2}{2} + \xi \end{bmatrix} + A_2 \begin{bmatrix} \frac{\xi^4}{4} - \frac{\xi^3}{3} - \frac{\xi^2}{2} + \xi & -\frac{\xi^4}{4} - \frac{\xi^3}{3} + \frac{\xi^2}{2} + \xi \\ -\frac{\xi^4}{4} - \frac{\xi^3}{3} + \frac{\xi^2}{2} + \xi & \frac{\xi^4}{4} + \xi^3 + \frac{3\xi^2}{2} + \xi \end{bmatrix}]_{-1}^{1}$$

Therefore:

$$m = \begin{bmatrix} \frac{\rho L(3A_1+A_2)}{12} & \frac{\rho L(A_1+A_2)}{12} \\ \frac{\rho L(A_1+A_2)}{12} & \frac{\rho L(A_1+3A_2)}{12} \end{bmatrix}$$

5 Assignment 10.5

A uniform two-node bar element is allowed to move in a 3D space. The nodes have only translational d.o.f. What is the diagonal mass matrix of the element? Assigning half of the mas in a constant A element of L size without any rotations allowed we get the following diagonal mass matrix of the element in a 3D space:

$$m = \begin{bmatrix} \frac{\rho AL}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\rho AL}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\rho AL}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\rho AL}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\rho AL}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\rho AL}{2} \end{bmatrix}$$