Universitat Politècnica de Catalunya
Numerical Methods in Engineering
Computational Structural Mechanics and Dynamics

Dynamics

Assignment 10

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1 Question 10.1

1.1 Statement

In the dynamic system of slide 6, let $r(t)$ be a constant force F. What is the effect of F on the time-dependent displacement $u(t)$ and the natural frequency of vibration of the system?

1.2 Solution

The equation in question is the following:

$$m\ddot{u} + ku = F \tag{1}$$

To solve it we must first solve the homogeneous equation:

$$m\ddot{u}_h + ku_h = 0 \tag{2}$$

There are many methods to solve this but it's easy to see that the function proportional to its derivative is the sine wave:

$$u_h(t) = A\sin(\omega t + \phi) \tag{3}$$

By putting ω into 2 we can see the parameters:

- $\omega = \sqrt{k/m} = 2\pi f$ is the angular speed, proportional to the frequency.
- A is the amplitude, dependent on initial conditions.
- ϕ is the phase, also dependent on initial conditions.

Now we must solve the particular equation. It's clear that a constant function $u_p(t) = a$ will fit the equation:

$$\begin{aligned} m\ddot{u}_p + ku_p &= F \\ ku_p &= F \end{aligned}$$

Hence our particular equation looks like:

$$u_p(t) = \frac{F}{k} \tag{4}$$

Since this is simply an offset we can label it $u_0 := F/k$. Now the solution to the system is the superposition of u_h and u_p. It therefore looks like:

$$u(t) = u_0 + A\sin(\omega t + \phi) \tag{5}$$

1.3 Conclusion

From equation 5 we see that F is not linked to the frequency $\omega = 2\pi f$. Instead it simply acts as an offset. The effect of F can be completely removed by working on a different base. If we work with $x(t) = u(t) - u_0$ the equation can be expressed as:

$$x(t) = A\sin(\omega t + \phi) \tag{6}$$

2 Question 10.2

2.1 Statement

A weight whose mass is m is placed at the middle of a uniform axial bar of length L that is clamped at both ends. The mass of the bar may be neglected. Estimate the natural frequency of vibration in terms of m, L, E and A. Suggestion: First determine the effective k.

2.2 Solution

We will start of with a formula for deflection in clamped bars:

$$u = \frac{FL^3}{192EI} \qquad @\ x = \frac{L}{2} \tag{7}$$

We need the inertia of the bar. For this we need to know its cross-section shape. Since it is unspecified, I decided to go for a circular area. The inertia will be:

$$I = \frac{\pi R^4}{4} = \frac{A^2}{4\pi} \tag{8}$$

And substitute into the equation:

$$u = \frac{\pi FL^3}{48EA^2} \qquad @\ x = \frac{L}{2} \tag{9}$$

We can now move on to determining its effective stiffness coefficient.

$$k = \frac{dF}{du} = \frac{48EA^2}{\pi L^3} \tag{10}$$

With this we can calculate the natural frequency of vibration:

$$\omega = \sqrt{\frac{k}{m}} = \frac{4A}{L}\sqrt{\frac{3E}{\pi mL}} \tag{11}$$

3 Question 10.3

3.1 Statement

Use the expression on slide 18 to derive the mass matrix of slide 17.

Note: there are no equations in slide 18 nor matrices in 17. I solved it assuming this refered to slides 20 and 19.

3.2 Solution

The equation in question is

$$\boldsymbol{M} = \int_\Omega \rho \boldsymbol{N}^T \boldsymbol{N} dV \tag{12}$$

In index notation this looks like

$$m_{ij} = \int_\Omega \rho N_i N_j dV \tag{13}$$

We must assume constant density and cross section:

$$\begin{aligned} m_{ij} &= \int_0^1 \rho N_i N_j A \, dx \\ &= \rho A \int_0^L N_i N_j \, dx \\ &= \rho A L \int_0^1 N_i N_j d\xi \end{aligned}$$

Hence the equation becomes:

$$\boldsymbol{M} = \rho A L \int_0^1 \begin{bmatrix} N_1^2 & N_1 N_2 \\ N_1 N_2 & N_2^2 \end{bmatrix} d\xi \tag{14}$$

This is a good point to introduce the shape functions:

$$N_1 = 1 - \xi \tag{15}$$

$$N_2 = \xi \tag{16}$$

And the integrals of the relevant combinations:

$$\int_0^1 N_1^2 d\xi = \int_0^1 (1 - 2\xi + \xi^2) \, d\xi = \left[\xi - \xi^2 + \frac{\xi^3}{3} \right]_0^1 = \frac{1}{3} \tag{17}$$

$$\int_0^1 N_1 N_2 d\xi = \int_0^1 (\xi - \xi^2) d\xi = \left[\frac{\xi}{2} - \frac{\xi^3}{3} \right]_0^1 = \frac{1}{6} \tag{18}$$

$$\int_0^1 N_2^2 d\xi = \int_0^1 \xi^2 \, d\xi = \left[\frac{\xi^3}{3} \right]_0^1 = \frac{1}{3} \tag{19}$$

Hence the mass matrix turns out

$$\boldsymbol{M} = \frac{\rho A L}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{20}$$

which is equivalent to the one found in slide 19.

□

4 Question 10.4

4.1 Statement

Obtain also the mass matrix of a two-node, linear displacement element with a variable cross-sectional area that varies from A1 to A2.

4.2 Solution

We can start defining the tapering area:

$$A = N_1 A_1 + N_2 A_2 = N_k A_k \tag{21}$$

Let's start as before:

$$\begin{aligned} m_{ij} &= \int_\Omega \rho N_i N_j dV \\ &= \rho \int_0^1 N_i N_j \frac{dV}{d\xi} d\xi \\ &= \rho \int_0^1 N_i N_j \frac{dV}{d\xi} d\xi \\ &= \rho \int_0^1 N_i N_j L A d\xi \\ &= \rho L A_k \int_0^1 N_i N_j N_k d\xi \end{aligned}$$

Hence:

$$\boldsymbol{M} = \rho L \left(A_1 \int_0^1 \begin{bmatrix} N_1^3 & N_1^2 N_2 \\ N_1^2 N_2 & N_1 N_2^2 \end{bmatrix} d\xi + A_2 \int_0^1 \begin{bmatrix} N_1^2 N_2 & N_1 N_2^2 \\ N_1 N_2^2 & N_2^3 \end{bmatrix} d\xi \right) \tag{22}$$

Here are the integrals:

$$\int_0^1 N_1^3 d\xi = \frac{1}{4} \qquad \int_0^1 N_1^2 N_2 d\xi = \frac{1}{12}$$

$$\int_0^1 N_2^3 d\xi = \frac{1}{4} \qquad \int_0^1 N_1 N_2^2 d\xi = \frac{1}{12}$$

The equation becomes:

$$\boldsymbol{M} = \rho L \left(A_1 \begin{bmatrix} 1/4 & 1/12 \\ 1/12 & 1/12 \end{bmatrix} + A_2 \begin{bmatrix} 1/12 & 1/12 \\ 1/12 & 1/4 \end{bmatrix} \right) = \frac{\rho L}{12} \left(A_1 \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} + A_2 \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \right) \tag{23}$$

Hence, the mass matrix is:

$$\boldsymbol{M} = \frac{\rho L}{12} \begin{bmatrix} 3A_1 + A_2 & A_1 + A_2 \\ A_1 + A_2 & A_1 + 3A_2 \end{bmatrix} \tag{24}$$

We can check that for $A := A_1 = A_2$ we recover the constant-area matrix:

$$\boldsymbol{M} = \frac{\rho L A}{12} \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} = \frac{\rho L A}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{25}$$

5 Question 10.5

5.1 Statement

A uniform two-node bar element is allowed to move in a 3D space. The nodes have only translational d.o.f. What is the diagonal mass matrix of the element?

5.2 Solution

We can get the solution straight from slide 16:

$$\boldsymbol{M} = \frac{\rho A L}{2}\boldsymbol{I}_6 = \frac{\rho A L}{2}\begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix} \tag{26}$$