Computational Structural Mechanics and Dynamics
Assignment 10 - Structural Dynamics

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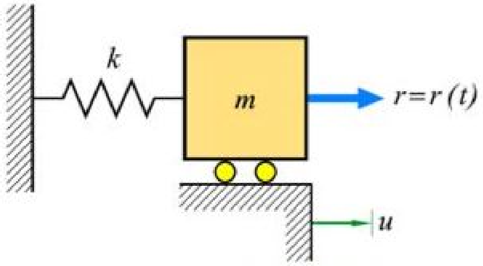
1 Question 1

In the dynamic system below, let r(t) be a constant force F. What is the effect of F on the time-dependent displacement u(t) and the natrual frequency of vibration of the system?



Considering r(t) to be a constant load F, the second order linear ODE becomes takes the form:

$$ku_{(t)} + m\ddot{u}_{(t)} = F \tag{1}$$

The solution for a non-homogeneous ODE of this kind is given by:

$$u_{(t)} = u_h + u_p \tag{2}$$

Where u_h is the homogeneous solution and u_p is the particular solution. For this particular ODE:

$$u_h = A\, sin\left(\sqrt{\frac{k}{m}}t\right) + B\, cos\left(\sqrt{\frac{k}{m}}t\right) \tag{3}$$

$$u_p = \frac{F}{k} \tag{4}$$

Considering $\omega = \sqrt{k/m}$ and replacing (3) and (4) into (2) yields:

$$u_{(t)} = A\, sin(\omega t) + B\, cos(\omega t) + \frac{F}{k} \tag{5}$$

At the instant $t = 0$ when F is applied, the system is at rest. Therefore, the initial conditions for the ODE are assumed to be:

$$u_0 = \dot{u}_0 = 0 \tag{6}$$

Replacing these initial conditions in the solution and its first derivative respectively, the values for the integration constants are obtained and the equation is solved.

$$A = 0 \qquad B = -\frac{F}{k} \tag{7}$$

$$\boxed{u_{(t)} = \frac{F}{k}\left[1 - cos(\omega t)\right]} \tag{8}$$

In conclusion, F does not affect the natural vibration frequency of the system ω and does not produce a phase shift. It does however affect the amplitude of the vibration and the range of movement. The system will not oscillate harmonically between $[-\overline{u}, \overline{u}]$ as a sine function, but between $[0, 2\frac{F}{k}]$ as a vertically displaced cosine function.

2 Question 2

A weight whose mass is m is placed at the middle of a uniform axial bar of length L that is clamped at both ends. The mass of the bar may be neglected. Estimate the natural frequency of vibration in terms of m, L, E and A.

Since the bar is clamped at both ends, it is possible to simplify the structure into a single D.O.F. structural system. The degree of freedom will be associated to the vertical displacement at the center of the bar where the weight is placed. For this purpose, we will use the analytical expression for the maximum deflection of a beam that is clamped at both ends and is subjected to a point load in the span-center, which is given by:

$$\delta = \frac{FL^3}{192EI} \tag{9}$$

Where:

- δ is the vertical displacement, which from now on will be referred to as u.
- $F = mg$ is the applied force, which in this case corresponds to the concentrated weight.
- L is the length of the bar.
- E is the Young's modulus of the material.
- I is the moment of inertia of the bar. In this case, assuming that the cross section of the bar is a square of side length $b = h = \sqrt{A}$, the moment of inertia takes the form $I = \frac{bh^3}{12} = \frac{A^2}{12}$

Rewriting the equation in the form $Ku = F$ yields:

$$\left(16\frac{EA^2}{L^3}\right)u = mg \tag{10}$$

$$K = 16\frac{EA^2}{L^3} \tag{11}$$

Therefore, the natural frequency of vibration ω of the system takes the form:

$$\boxed{\omega = \sqrt{\frac{K}{m}} = \frac{4A}{L}\sqrt{\frac{E}{mL}}} \tag{12}$$

3 Question 3

Use the expression $m = \int N^T N \rho dV$ to derive the following mass matrix

$$\mathbf{m} = \begin{bmatrix} \rho AL/3 & \rho AL/6 \\ \rho AL/6 & \rho AL/3 \end{bmatrix} \tag{13}$$

Assuming constant density ρ and cross section A, the expression for the mass matrix takes the form:

$$m = \rho A \int_0^L N^T N dx \tag{14}$$

Where N is vector containing the shape functions in the form $N = [N_1 \; N_2]$. The shape functions for a linear two-node 1D element may be written as:

$$N_1(x) = \frac{L-x}{L} \qquad N_2(x) = \frac{x}{L} \tag{15}$$

Therefore, the mass matrix takes the form:

$$m = \rho A \int_0^L \begin{bmatrix} N_1^2 & N_1N_2 \\ N_1N_2 & N_2^2 \end{bmatrix} dx = \frac{\rho A}{L^2} \int_0^L \begin{bmatrix} (L-x)^2 & (L-x)x \\ (L-x)x & x^2 \end{bmatrix} dx \tag{16}$$

$$\boxed{m = \frac{\rho A L}{3} \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}} \tag{17}$$

4 Question 4

Obtain also the mass matrix of a two-node, linear displacement element with a variable cross-sectional area that varies from A_1 to A_2.

Using the same shape functions proposed in Equation (15), we obtain a linear interpolation for the cross section of linear displacement element in the form:

$$A(x) = A_1N_1(x) + A_2N_2(x) \tag{18}$$

As $A(x)$ can no longer be treated as a constant, Equation (6) now becomes:

$$m = \rho \int_0^L N^T N A(x) dx \tag{19}$$

Separating the terms associated to A_1 and A_2 yields:

$$m = \rho A_1 \int_0^L \begin{bmatrix} N_1^3 & N_1^2N_2 \\ N_1^2N_2 & N_1N_2^2 \end{bmatrix} dx + \rho A_2 \int_0^L \begin{bmatrix} N_1^2N_2 & N_1N_2^2 \\ N_1N_2^2 & N_2^3 \end{bmatrix} dx \tag{20}$$

Following the same procedure proposed in equations (16) and (17), the mass matrix takes the form:

$$\boxed{m = \frac{\rho L}{4} \left(A_1 \begin{bmatrix} 1 & 1/3 \\ 1/3 & 1/3 \end{bmatrix} + A_2 \begin{bmatrix} 1/3 & 1/3 \\ 1/3 & 1 \end{bmatrix} \right)} \tag{21}$$

5 Question 5

A uniform two-node bar element is allowed to move in a 3D space. The nodes have only translational d.o.f. What is the diagonal mass matrix of the element?

The element has six degrees of freedom corresponding to the three displacements of each node. Since the element is uniform, the mass will be distributed evenly between the two nodes. Therefore, the diagonal (lumped) mass matrix will take the form:

$$\boxed{m = \frac{\rho A L}{2} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 1 & 0 \\ sym. & & & & & 1 \end{bmatrix}} \tag{22}$$