Computational Structural Mechanics and Dynamics
MSc in Computational Mechanics
Universitat Politecnica de Catalunya

Assignment 10
Dynamics

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Contents

1. Abstract

In this report is shown the solution of dynamics problems, stated in the assignment sheet on the CIMNE virtual center.

2. Question 1

In the dynamic system of slide 6, let r(t) be a constant force F. What is the effect of F on the time-dependent displacement u(t) and the natural frequency of vibration of the system?

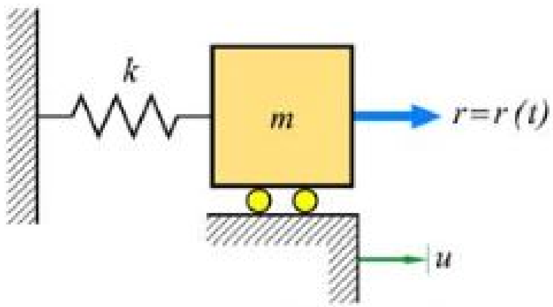

Figure 2.1: Dynamic system

As in dynamics problems, it has to be considered the Newton's second law about forces balance $F = ma$. Considering $u(t)$ the displacement and the force constant, the Newton's second law takes the form of:

$$m\ddot{u} + ku = F \tag{2.1}$$

The solution of this non-homogeneous ordinary differential equation will be composed by the general solution plus the particular one. In order to find the general solution, it has to be considered the free clamped system, so with $F = 0$.

$$m\ddot{u} + ku = 0 \tag{2.2}$$

The general solution will be:

$$u(t) = Asin(\omega t + \phi) + Bcos(\omega t + \phi) \tag{2.3}$$

So assuming initial conditions $u(t = 0) = 0$ and no initial phase ($\phi = 0$), the solution for the force equation is:

$$u(t) = Bcos(\omega t) \tag{2.4}$$

where A is the amplitude of motion and ω is the natural frequency of vibration (rad/s) and is equal to $\sqrt{k/m}$. It can be seen from here that F will not affect the natural frequency of vibration ω as it only depends on the values of k and m.

The particular solution will be computed considering u constant but the force different from 0, so the equation will lead us to:

$$ku_p = F \;\rightarrow\; u_p = F/k \tag{2.5}$$

My equation now will be:

$$u(t) = Bcos(\sqrt{k/m}t) + F/k \tag{2.6}$$

in order to find B, it has to be considered the initial condition and B will be then equal to $-F/k$. So the final force equation will be:

$$u(t) = \frac{F}{k}\left(1 - cos\left(t\sqrt{\frac{k}{m}}\right)\right) \tag{2.7}$$

in which can be appreciated the effect of F in the displacements.

3. Question 2

A weight whose mass is m is placed at the middle of a uniform axial bar of length L that is clamped at both ends. The mass of the bar may be neglected. Estimate the natural frequency of vibration in terms of m, L, E and A. Suggestion: First determine the effective k.

The only displacements that will be permitted will be the one along the negative y-direction as the beam is clamped in both the extremities and it is a gravity problem: the only force acting is the gravity one, in the middle of the beam.

Remembering that the natural frequency of vibration is $\sqrt{k/m}$ it is needed to find k. Remembering that:

$$ku = F \ \rightarrow \ k = F/u \tag{3.1}$$

In order to find k, it has to be considered the maximum vertical displacement that will be located in the middle of the beam where the load is applied. The displacement will be equal to:

$$u = \frac{mgL^3}{192EI} \tag{3.2}$$

in which E is known as is the Young modulus of the beam and the inertia I has to be computed. Choosing a circular section, the inertia will be equal to:

$$I = \pi d^4/64 \tag{3.3}$$

the Area of a circular section is $A = \pi r^2$ so the inertia, written in function of A, will be:

$$I = A^2/(4\pi) \tag{3.4}$$

Plugging 3.4 in 3.2 and then in 3.1 it is found the relation of k and so, plugging it in the expression of the natural frequency of the system, the following relation is obtained:

$$\omega = \sqrt{\frac{48EA^2}{mL^3\pi}} = \frac{4A}{L}\sqrt{\frac{3E}{L\pi m}} \tag{3.5}$$

4. Question 3

Use the expression on slide 20 to derive the mass matrix of slide 19.
The expression taken into account is the consistent element mass matrix defined as following:

$$\mathbf{M} = \int \mathbf{N}^T \mathbf{N} \rho dV \tag{4.1}$$

in which $\mathbf{N}$ are defined as:

$$\begin{aligned} N_1 &= 1 - x/L \\ N_2 &= x/L \end{aligned} \tag{4.2}$$

and are the shape functions of my element. Considering density and cross section as constants, the consistent element mass matrix takes the form of:

$$\mathbf{M} = \rho A \int_0^L \mathbf{N}^T \mathbf{N} dx \tag{4.3}$$

Solving the system will lead to:

$$\mathbf{M} = \rho A \int_0^L \begin{bmatrix} N_1^2 & N_1 N_2 \\ N_1 N_2 & N_2^2 \end{bmatrix} dx \tag{4.4}$$

Computing each integral with the shape functions defined in 4.2:

$$\begin{aligned} \int_0^L N_1^2 dx &= \left[x + \frac{x^3}{3L^2} - \frac{2x^2}{2L} \right]_0^L = \frac{L}{3} \\ \int_0^L N_2^2 dx &= \left[\frac{x^3}{3L^2} \right]_0^L = \frac{L}{3} \\ \int_0^L N_1 N_2 dx &= \left[-\frac{x^3}{3L^2} + \frac{x^2}{2L} \right]_0^L = \frac{L}{6} \end{aligned} \tag{4.5}$$

Substituting everything in 4.4 it will be recovered the expression asked, in fact:

$$\mathbf{M} = \rho A \begin{bmatrix} \frac{L}{3} & \frac{L}{6} \\ \frac{L}{6} & \frac{L}{3} \end{bmatrix} = \begin{bmatrix} \frac{\rho A L}{3} & \frac{\rho A L}{6} \\ \frac{\rho A L}{6} & \frac{\rho A L}{3} \end{bmatrix} \tag{4.6}$$

5. Question 4

Obtain also the mass matrix of a two-node, linear displacement element with a variable cross-sectional area that varies from A_1 to A_2.

The variation of the cross section will depends on x. Hence it can be described in function of x, using the same shape functions defined in the previous point, 4.2. Describing the area, the following expression yields:

$$A(x) = A_1 N_1(x) + A_2 N_2(x) \tag{5.1}$$

Computing as previously done, but considering the linear variation of the cross section:

$$\mathbf{M} = \rho \left(A_1 \int_0^L \begin{bmatrix} N_1^3 & N_1^2 N_2 \\ N_1^2 N_2 & N_2^2 N_1 \end{bmatrix} dx + A_2 \int_0^L \begin{bmatrix} N_1^2 N_2 & N_1 N_2^2 \\ N_1 N_2^2 & N_2^3 \end{bmatrix} dx \right) \tag{5.2}$$

Computing the integrals as before:

$$\begin{aligned} \int_0^L N_1^3 dx &= \left[x + \frac{3x^3}{3L^2} - \frac{x^4}{4L^3} - \frac{3x^2}{2L} \right]_0^L = \frac{L}{4} \\ \int_0^L N_2^3 dx &= \left[\frac{x^4}{4L^3} \right]_0^L = \frac{L}{4} \\ \int_0^L N_1^2 N_2 dx &= \left[\frac{x^2}{2L} + \frac{x^4}{4L^3} - \frac{2x^3}{3L^2} \right]_0^L = \frac{L}{12} \\ \int_0^L N_1 N_2^2 dx &= \left[\frac{x^3}{3L^2} - \frac{x^4}{4L^3} \right]_0^L = \frac{L}{12} \end{aligned} \tag{5.3}$$

Hence the consistent element mass matrix will be:

$$\mathbf{M} = \rho L \left(A_1 \begin{bmatrix} \frac{1}{4} & \frac{1}{12} \\ \frac{1}{12} & \frac{1}{12} \end{bmatrix} + A_2 \begin{bmatrix} \frac{1}{12} & \frac{1}{12} \\ \frac{1}{12} & \frac{1}{4} \end{bmatrix} \right) \tag{5.4}$$

6. Question 5

A uniform two-node bar element is allowed to move in a 3D space. The nodes have only translational d.o.f. What is the diagonal mass matrix of the element?

The mass matrix corresponding to a two-noded bar element will be a $6x6$ matrix, the three directions each node. Remembering that the terms on the diagonal M_{ii} refer to the uniaxial displacements and the terms M_{ij} for $i \neq j$ refer to rotations. If the nodes are allowed to have only translational degree of freedom, the resultant mass matrix will have only the terms on the diagonal $M_{ii} = (\rho AL/2)$.

Explicitly the mass matrix is:

$$\mathbf{M} = \frac{\rho LA}{2} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{6.1}$$