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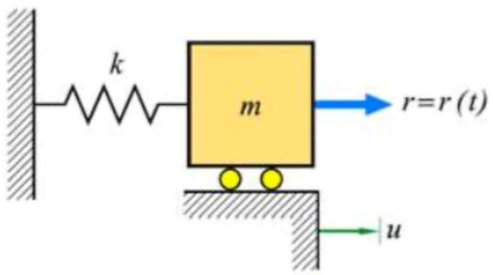
Assignment 10- Dynamics

Question 1)

In the dynamic system of slide 6, let r(t) be a constant force F. What is the effect of F on the time-dependent displacement u(t) and the natural frequency of vibration of the system?



I have to consider the second law of Newton about forces balance, that is:

$$F = m * a$$

In this case the force is constant and if we consider $u(t)$ the displacement the equation will be:

$$F = m\ddot{u} + ku$$

Now I obtained the solution for a non-homogeneous ordinary differential equation ODE, which include a general solution and a particular one.

$$0 = m\ddot{u} + ku$$

General solution:

$$u(t) = Asin(wt + \phi) + Bcos(wt + \phi)$$

Where A is the amplitude of motion; $w = \sqrt{\frac{k}{m}}$ is the natural frequency of vibration.

Calculation of the solution for the initial condition $u(t = 0) = 0, \phi = 0$:
General solution:

$$u(t) = Bcos(wt)$$

Particular solution:

$$F = u_p K \quad \rightarrow u_p = \frac{F}{k}$$

Total Equation:

$$u(t) = Bcos(wt) + u_p = Bcos(wt) + \frac{F}{k}$$

Now I have to find the value of B.

$$initial\ condition\ u(t=0) = 0 : \ A = 0 \quad and\ B = -\frac{F}{k}$$

So the final force equation will be:

$$u(t) = \frac{F}{k}(1 - \cos(wt)) = \frac{F}{k}(1 - \cos\left(t * \left(\sqrt{\frac{k}{m}}\right)\right)$$

Observation: F does not produce a phase shift and doesn't affect the natural vibration frequency of the system ω. It affects the amplitude of the vibration and the range of movement.

Question 2)

A weight whose mass is m is placed at the middle of a uniform axial bar of length L that is clamped at both ends. The mass of the bar may be neglected. Estimate the natural frequency of vibration in terms of m, L, E and A. Suggestion: First determine the effective k.

The natural frequency of vibration is $w = \sqrt{K/m}$, so I have to find k. The fist equation was:

$$F = ku$$

So K will be:

$$K = \frac{F}{u}$$

We have to consider the maximum vertical displacement in the middle of the beam (point of the load application). This displacement is:

$$u = \frac{F}{EI} * \frac{l^3}{192}$$

Where:

- E in the Young modulus,
- F=m*g
- I the inertia that is equal to $I = \pi * \frac{d^4}{12} = \frac{A^2}{4\pi}$

The expression of the natural frequency will be:

$$w = \sqrt{K/m} = \frac{4A}{L} * \sqrt{\frac{E}{mL}}$$

Question 3

Use the expression $\boldsymbol{m} = \int \boldsymbol{N}^{\boldsymbol{T}} * \boldsymbol{N} * \boldsymbol{\rho} \boldsymbol{dV}$ *to derive the following mass matrix.*

It is possible to define N into two shape function, which are:

$$N_1 = 1 - \frac{x}{L}$$

$$N_2 = \frac{x}{L}$$

The density and the cross section are constant, so it is possible to writhe the consistent element mass matrix as following:

$$M = \rho A \int_0^L N^T N dx$$

Where

$$N^T N = \begin{bmatrix} N_1^2 & N_1 N_2 \\ N_1 N_2 & N_2^2 \end{bmatrix}$$

And so:

$$M = \rho A \int_0^L N^T N dx = \rho A \int_0^L \begin{bmatrix} N_1^2 & N_1 N_2 \\ N_1 N_2 & N_2^2 \end{bmatrix} dx = \frac{\rho A}{L^2} \int_0^L \begin{bmatrix} (l-x)^2 & (l-x)x \\ (l-x)x & x^2 \end{bmatrix} dx =$$

$$M = \rho A \begin{bmatrix} \frac{L}{3} & \frac{L}{6} \\ \frac{L}{6} & \frac{L}{3} \end{bmatrix}$$

Question 4)

Obtain also the mass matrix of a two-node, linear displacement element with a variable cross-sectional area that varies from A1 to A2.

Taking into account the two shape functions $N_{1,}N_2$, of the last question I obtain a linear interpolation for
the cross section of linear displacement element:

$$A(x) = A_1 * N_1(x) + A_2 * N_2(x)$$

Now I'm considering the variation of the cross section, so I can calculate the value of M as:

$$M = \rho(A_1 \int_0^L \begin{bmatrix} N_1^3 & N_1^2 N_2 \\ N_1^2 N_2 & N_1 N_2^2 \end{bmatrix} dx + A_2 \int_0^L \begin{bmatrix} N_1^2 N_2 & N_1\ N_2^2 \\ N_1\ N_2^2 & N_2^3 \end{bmatrix} dx)$$

Computing the integral I obtained:

$$\int_0^L N_1^3 dx = \frac{L}{4}$$

$$\int_0^L N_2^3 dx = \frac{L}{4}$$

$$\int_0^L N_1^2 N_2\ dx = \frac{L}{12}$$

$$\int_0^L N_1\ N_2^2 dx = \frac{L}{12}$$

$$M = \rho L(A_1 \begin{bmatrix} \frac{1}{4} & \frac{1}{12} \\ \frac{1}{12} & \frac{1}{12} \end{bmatrix} + A_2 \begin{bmatrix} \frac{1}{12} & \frac{1}{12} \\ \frac{1}{12} & \frac{1}{4} \end{bmatrix})$$

Question 5)

A uniform two-node bar element is allowed to move in a 3D space. The nodes have only translational d.o.f. What is the diagonal mass matrix of the element?

The element bar has 6 degrees of freedom (3 displacement of each node) and the mass matrix correspond to a two-nodal bar element is 6x6. The mass matrix has only term on diagonal if the nodes can only have translational degree of freedom:

$$M = \frac{LA\rho}{2}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$