Universitat Politècnica de Catalunya
Numerical Methods in Engineering
Computational Solid Mechanics and Dynamics

Variational Formulation Addendum

Assignment 2 extra

Eduard Gómez
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1 Statement

a) Derive the stiffness matrix for a tapered bar element in which the cross section area varies linearly along the element length:

$$A = (1 - \xi_1)A_1 + \xi A_2 \tag{1}$$

where A_1 and A_2 are the areas at the end nodes, and ξ is the natural dimensionless coordinate for a bar member. Show that yields to the same answer that of a stiffness of a constant area bar with cross section $A = \frac{1}{2}(A_1 + A_2)$.

b) Find the consistent load vector f^e for a bar of constant area A subject to a force $q = \rho g A(\xi)$ which $A(\xi)$ varies according to question a) and ρ, g are constants. Check the cases $A_1 = A_2$, and$A_2 = 0$.

c) Find the consistent load vector f^e if the bar is subjected to a concentrated axial force Q at a distance $x = a$ from its left end. Consider $q(x) = Q\delta(x - a)$ in which $\delta(x - a)$ is the one-dimensional Dirac delta function at $x = a$. Check the results for the relevant cases of a.

2 Solution

We'll start by stating the balance on a slice of width Δx of the element:

$$-\sigma_1 A_1 + \sigma_2 A_2 + q\Delta x = 0 \tag{2}$$

Replacing for an infinitesimal slice:

$$-\sigma A + \left(\sigma A + \frac{\partial(\sigma A)}{\partial x}\,dx\right) + q\,dx = 0 \tag{3}$$

Hence:

$$\frac{d(\sigma A)}{dx} + q = 0$$

$$\sigma\frac{dA}{dx} + A\frac{d\sigma}{dx} + q = 0$$

Including the constitutive equation $\sigma = E\dfrac{du}{dx}$:

$$E\frac{du}{dx}\frac{dA}{dx} + EA\frac{d^2u}{dx^2} + q = 0 \tag{4}$$

Hence we reach the strong form of the problem:

$$-EA\frac{d^2u}{dx^2} - E\frac{du}{dx}\frac{dA}{dx} = q \tag{5}$$

We'll multiply the test function and integrate:

$$-E\int_{x_1}^{x_2} A\frac{d^2u}{dx^2}v\,dx - E\int_{x_1}^{x_2}\frac{du}{dx}\frac{dA}{dx}v\,dx = \int_{x_1}^{x_2} qv\,dx \tag{6}$$

Using the chain rule on the first term:

$$-E\int_{x_1}^{x_2} A\frac{d^2u}{dx^2}v\,dx = -E\int_{x_1}^{x_2} A\frac{d}{dx}\left(v\frac{du}{dx}\right)dx + E\int_{x_1}^{x_2} A\frac{du}{dx}\frac{dv}{dx}\,dx \tag{7}$$

Using the chain rule again, this time on the first term of the right hand side, we obtain:

$$-E\int_{x_1}^{x_2} A\frac{d^2u}{dx^2}v\,dx = -E\int_{x_1}^{x_2} \frac{d}{dx}\left(Av\frac{du}{dx}\right)\,dx + E\int_{x_1}^{x_2} \frac{dA}{dx}v\frac{du}{dx}\,dx + E\int_{x_1}^{x_2} A\frac{du}{dx}\frac{dv}{dx}\,dx \tag{8}$$

Using the divergence theorem in 1D, also knwon as Gauss' theorem:

$$-E\int_{x_1}^{x_2} A\frac{d^2u}{dx^2}v\,dx = -\left[Av\frac{du}{dx}\right]_{x_1}^{x_2} + E\int_{x_1}^{x_2} \frac{dA}{dx}v\frac{du}{dx}\,dx + E\int_{x_1}^{x_2} A\frac{du}{dx}\frac{dv}{dx}\,dx \tag{9}$$

Putting this back in equation 6:

$$\left(-\left[Av\frac{du}{dx}\right]_{x_1}^{x_2} + E\int_{x_1}^{x_2} \frac{dA}{dx}v\frac{du}{dx}\,dx + E\int_{x_1}^{x_2} A\frac{du}{dx}\frac{dv}{dx}\,dx\right) - E\int_{x_1}^{x_2} \frac{du}{dx}\frac{dA}{dx}v\,dx = \int_{x_1}^{x_2} qv\,dx \tag{10}$$

The second and last terms on the left hand side cancel out. Rearranging it it becomes:

$$E\int_{x_1}^{x_2} A\frac{du}{dx}\frac{dv}{dx}\,dx = \int_{x_1}^{x_2} qv\,dx + \left[Av\frac{du}{dx}\right]_{x_1}^{x_2} \tag{11}$$

Since this is for an arbitrary element, the flux is not prescribed. We reach our last step before introducing the shape functions.

$$E\int_{x_1}^{x_2} A\frac{du}{dx}\frac{dv}{dx}\,dx = \int_{x_1}^{x_2} qv\,dx \tag{12}$$

We will now make the following replacements:

$$u(x) = \textstyle\sum_{j=1}^{p} N_j(x)u(x_j) \qquad \frac{du}{dx} = \textstyle\sum_{j=1}^{p} \frac{dN_j}{dx}u(x_j) \qquad v_i(x) = N_i(x) \tag{13}$$

where p is the order of discretization. Replacing yields:

$$\left(E\int_{x_1}^{x_2} A\frac{dN_i}{dx}\frac{dN_j}{dx}\,dx\right)u_j = \int_{x_1}^{x_2} qN_i\,dx \qquad i,j = 1,2...p \tag{14}$$

The previous equation can be expressed in matrix form:

$$\boldsymbol{KU} = \boldsymbol{F} \tag{15}$$

where

$$K_{ij} = E\int_{x_1}^{x_2} A\frac{dN_i}{dx}\frac{dN_j}{dx}\,dx \tag{16}$$

$$U_i = u(x_i) \tag{17}$$

$$F_i = \int_{x_1}^{x_2} qN_i\,dx \tag{18}$$

Since $A = A(\xi)$, for $\xi = [0,1]$, we have to do a change of variables. We'll work in the ξ space since it works similar to isoparametric space. The matrices become:

$$K_{ij} = \frac{E}{h}\int_0^1 [(1-\xi_1)A_1 + \xi A_2]\,\frac{dN_i}{d\xi}\frac{dN_j}{d\xi}\,d\xi \tag{19}$$

$$U_i = u(x_i) \tag{20}$$

$$F_i = h\int_0^1 qN_i\,d\xi \tag{21}$$

We'll particularize for linear elements:

$$N_1(\xi) = 1 - \xi \qquad N_2(\xi) = \xi \tag{22}$$

Then we get, for the stiffness matrix:

$$K = \frac{E}{h}\frac{A_1 + A_2}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{23}$$

If we define the average area as $\bar{A}$:

$$K = \frac{E\bar{A}}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{24}$$

which is the same as a constant section bar of area $\bar{A}$. **This solves part(a).**

In the force vector we must substitute $q(x) = \rho g A(\xi)$. After integrating it becomes:

$$F = \frac{\rho g h}{6}\begin{bmatrix} 2A_1 + A_2 \\ A_1 + 2A_2 \end{bmatrix} \tag{25}$$

Of course for $A_1 = A_2 = A$ we recover the same force vector we had for constant area elements:

$$F = \frac{\rho g h A}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{qh}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{26}$$

And in the case $A_1 \neq A_2 = 0$ we retrieve:

$$F = \frac{\rho g h A_1}{6}\begin{bmatrix} 2 \\ 1 \end{bmatrix} \tag{27}$$

This solves part(b).

If, on the other hand, we have it so $q(x) = Q\delta(x-a)$, where $a \in [x_1, x_2]$; the integral looks like:

$$F_i = Q\int_{x_1}^{x_2} N_i \delta(x-a)dx \tag{28}$$

We must use the following property:

$$\int_{z_1}^{z_2} f(x)\delta(x-a)dx = f(a) \iff z_1 < a < z_2 \tag{29}$$

Applying this yields:

$$F_i = QN_i(\frac{a - x_1}{h}) \tag{30}$$

Therefore:

$$F = Q\begin{bmatrix} 1 - a/h \\ a/h \end{bmatrix} \tag{31}$$

We can see that for $a = 0$ we simply have the vector with an external load at the first node. For $a = h$ we have the load at the second load. Finally, for $a = \frac{h}{2}$ we have the same force vector as a uniformly distributed load $q^* = \frac{Q}{h}x$. **This solves part (c).**