Computational Structural Mechanics and Dynamics

As2_extra Variational formulation

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Assignment 2.2

A) Derive the stiffness for a tapered bar element in which the cross section area varies linearly along the element length:

$$\boldsymbol{A = A_i(1-\xi) + A_j\xi}$$

Where A_i and A_j are the areas at the end nodes, and ξ is the natural dimensionless coordinate for a bar member. Show that yields to the same answer that of a stiffness of a constant area bar with cross section

$$\boldsymbol{A = 1/2(A_i + A_j)}$$

[Answer]

First, the ξ coordinate is expressed in terms of the element coordinate x:

$$\xi = \frac{x}{l}$$

Then the unknown variables are expressed in function of ξ

$$N = \begin{bmatrix} N_1(\xi) \\ N_2(\xi) \end{bmatrix} = \begin{bmatrix} 1-\xi \\ \xi \end{bmatrix}$$

$$Ka = f$$

$$K_{ij}^e = \int_0^{l^e} \frac{dN_i^e}{dx} k \frac{dN_j^e}{dx} dx = \int_0^1 \frac{dN_i^e}{d\xi} \frac{EA}{l} \frac{dN_j^e}{d\xi} d\xi = \frac{E}{l} \int_0^1 A_i(1-\xi) + A_j\xi \frac{dN_i^e}{d\xi} \frac{dN_j^e}{d\xi} d\xi$$

$$= \frac{E}{l} \int_0^1 (A_i(1-\xi) + A_j\xi) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} d\xi = \frac{E}{l} \frac{A_i + A_j}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

If we set $A = 1/2(A_i + A_j)$, we obtain the

$$K_{ij}^e = \frac{E}{l} A \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Which is the same answer that of a stiffness of a constant area bar

$$A = 1/2(A_i + A_j)$$

B) Find the consistent load vector $\boldsymbol{f^e}$ for a bar of constant area A subject to a force $\boldsymbol{q(x) = \rho g A(\xi)}$ in which $A(\xi)$ varies according to question a) and $\boldsymbol{\rho, g}$ are constants. Check the case $\boldsymbol{A_i = A_j}$, and $\boldsymbol{A_j = 0}$.

[Answer]

$$f = \int_0^1 N \cdot q \cdot J^{-1} d\xi = \int_0^1 \begin{bmatrix} 1-\xi \\ \xi \end{bmatrix} \cdot \rho g (A_i(1-\xi) + A_j\xi) \cdot l \, d\xi$$

$$= \rho g l \int_0^1 A_i \begin{bmatrix} (1-\xi)^2 \\ \xi(1-\xi) \end{bmatrix} + A_j \begin{bmatrix} \xi(1-\xi) \\ \xi^2 \end{bmatrix} d\xi$$

$$= \rho g l (A_i \begin{bmatrix} \frac{1}{3} \\ \frac{1}{6} \end{bmatrix} + A_j \begin{bmatrix} \frac{1}{6} \\ \frac{1}{3} \end{bmatrix})$$

Case $A_i = A_j = A$,

$$f = \rho g l A \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = ql \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

Case $A_j = 0$,

$$f = \rho g l A_i \begin{bmatrix} \frac{1}{3} \\ \frac{1}{6} \end{bmatrix}$$

C) Find the consistent load vector $\boldsymbol{f^e}$ if the bar is subjected to a concentrated axial force $\boldsymbol{Q}$ at a distance $\boldsymbol{x = a}$ from its left end. Consider $\boldsymbol{q(x) = Q\delta(x - a)}$ in which $\boldsymbol{\delta(x-a)}$ is the one-dimensional Dirac's delta function at $\boldsymbol{x = a}$. Check the results for the relevant case of $\boldsymbol{a}$.

[Answer]

$$F_i = \int_0^l q\, N_i dx = Q \int_0^l \delta(x-a)\, N_i dx = Q \int_0^l \delta(x-a) \begin{bmatrix} 1 - \frac{x}{l} \\ \frac{x}{l} \end{bmatrix} dx$$

$$= Q \int_0^1 \delta(\xi l - a) \begin{bmatrix} 1-\xi \\ \xi \end{bmatrix} l d\xi$$

Since,

$$\int_{-\infty}^{\infty} \delta(x-a)\, dx = 1$$

$$\int_{z_1}^{z_2} f(x)\delta(x-a)\, dx = \begin{cases} 0, & a < z_1 \; or \; a > z_1, \quad we\; can\; not\; let\; x = a \\ f(a), & z_1 < a < z_2, \quad we\; can\; let\; x = a \end{cases}$$

We apply this yield:

$$F_i = QN_i\left(\frac{a}{l}\right)$$

So

$$F = Q \begin{bmatrix} 1 - \frac{a}{l} \\ \frac{a}{l} \end{bmatrix}$$

While $a = 0$, we have the vector with external load at Node 1. While $a = l$, we have the vector with external load at Node 2. While $a = l/2$, we have the same force vector as a uniform distributed load

$$q^* = Q\frac{x}{l}$$