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Assignment # 3

3.1 a) Plane stress

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{bmatrix} = \begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & (1-\nu)/2\end{bmatrix}\frac{E}{1-\nu^2}\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ 2\varepsilon_{xy}\end{bmatrix}$$

$$\sigma_{xx} = \frac{E}{1-\nu^2}\varepsilon_{xx} + \frac{E\nu}{1-\nu^2}\varepsilon_{yy}$$

$$\frac{E}{1-\nu^2} = \frac{E^*(1-\nu^*)}{(1+\nu^*)(1-2\nu^*)}$$

$$E^* = \frac{E(1+\nu^*)(1-2\nu^*)}{(1-\nu^*)(1-\nu^2)} \quad ①$$

plane strain

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & \frac{1-2\nu}{2}\end{bmatrix}\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ 2\varepsilon_{xy}\end{bmatrix}$$

$$\sigma_{xx} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\varepsilon_{xx} + \frac{E\nu}{(1+\nu)(1-2\nu)}\varepsilon_{yy}$$

$$\frac{E\nu}{1-\nu^2} = \frac{E^*\nu^*}{(1+\nu^*)(1-2\nu^*)}$$

$$E^* = \frac{E\nu(1+\nu^*)(1-2\nu^*)}{\nu^*(1-\nu^2)} \quad ②$$

equating the 2 equations

$$\frac{E(1+\nu^*)(1-2\nu^*)}{(1-\nu^*)(1-\nu^2)} = \frac{E\nu(1+\nu^*)(1-2\nu^*)}{\nu^*(1-\nu^2)}$$

$$\frac{1}{1-\nu^*} = \frac{\nu}{\nu^*} \qquad \nu = \frac{\nu^*}{1-\nu^*} \quad ③ \Rightarrow \boxed{\nu^* = \frac{\nu}{1+\nu}}$$

Substituting ③ in ①

$$E^* = \frac{E(1+\nu^*)(1-2\nu^*)}{(1-\nu^*)\left(1-\left(\frac{\nu^*}{1-\nu^*}\right)^2\right)} = \frac{E(1+\nu^*)(1-2\nu^*)}{(1-\nu^*)\left(1-\frac{\nu^*}{1-\nu^*}\right)\left(1+\frac{\nu^*}{1-\nu^*}\right)} = \frac{E(1+\nu^*)\cancel{(1-2\nu^*)}}{\cancel{(1-\nu^*)}\left(\frac{\cancel{1-2\nu^*}}{\cancel{1-\nu^*}}\right)\left(\frac{1}{1-\nu^*}\right)}$$

$$E^* = E(1+\nu^*)(1-\nu^*) = E(1-\nu^{*2}) \quad ④ \qquad \boxed{E^* = \frac{E(1+2\nu)}{(1+\nu)^2}}$$

$$E = \frac{E^*}{1-\nu^{*2}} \quad ④$$

if we replace E, ν with equations 3, 4 (E^*, ν^*) in the plane stress matrix, we get the plane strain matrix plane stress

plain strain

b) $\sigma_{xx} = \frac{E}{1-\nu^2}\varepsilon_{xx} + \frac{E\nu}{1-\nu^2}\varepsilon_{yy}$ $\Big\}$ $\sigma_{xx} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\varepsilon_{xx} + \frac{E\nu}{(1+\nu)(1-2\nu)}\varepsilon_{yy}$

$$\frac{E^*}{1-\nu^{*2}} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \qquad \frac{E^*\nu^*}{1-\nu^{*2}} = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

$$E^* = \frac{E(1-\nu)(1-\nu^{*2})}{(1+\nu)(1-2\nu)} \quad ① \qquad E^* = \frac{E\nu(1-\nu^{*2})}{\nu^*(1+\nu)(1-2\nu)} \quad ②$$

equating ① and ②

$$\frac{E(1-\nu)\cancel{(1-\nu^{*2})}}{\cancel{(1+\nu)(1-2\nu)}} = \frac{E\nu\cancel{(1-\nu^{*2})}}{\nu^*\cancel{(1+\nu)(1-2\nu)}}$$

$$1-\nu = \frac{\nu}{\nu^*} \Rightarrow \nu = \frac{\nu^*}{1+\nu^*} \quad ③ \qquad \boxed{\nu^* = \frac{\nu}{1-\nu}}$$

substituting ③ in ①

$$E^* = \frac{E\left(1-\frac{\nu^*}{1+\nu^*}\right)(1-\nu^{*2})}{\left(1+\frac{\nu^*}{1+\nu^*}\right)\left(1-\frac{2\nu^*}{1+\nu^*}\right)} = \frac{E\left(\frac{1}{\cancel{1+\nu^*}}\right)\cancel{(1-\nu^{*2})}}{\left(\frac{1+2\nu^*}{1+\nu^*}\right)\left(\frac{1-\nu^*}{1+\nu^*}\right)}$$

$$E^* = \frac{E(1+\nu^*)^2}{1+2\nu^*} \qquad E = \frac{E^*(1+2\nu^*)}{(1+\nu^*)^2} \quad ④$$

$$E^* = \frac{E\left(1+\frac{\nu}{1-\nu}\right)^2}{1+\frac{2\nu}{1-\nu}} = \frac{\frac{E}{(1-\nu)^2}}{\frac{1+\nu}{1-\nu}} = \frac{E}{1-\nu^2}$$

$$\boxed{E^* = \frac{E}{1-\nu^2}}$$

substituting ③, ④ in the plane strain Constitutive matrix, will lead to the plane stress Constitutive matrix

3.2 a) $\lambda = \dfrac{E\nu}{(1+\nu)(1-2\nu)}$, $\mu = \dfrac{E}{(1+\nu)2}$

$$E = \frac{\lambda(1+\nu)(1-2\nu)}{\nu} \quad ①$$

substitute in μ equation

$$\mu = \frac{\lambda\cancel{(1+\nu)}(1-2\nu)}{2\nu\cancel{(1+\nu)}}$$

$$2\mu\nu = \lambda - 2\lambda\nu$$

$$2\mu\nu + 2\lambda\nu = \lambda$$

$$\boxed{\nu = \frac{\lambda}{2(\mu+\lambda)}} \quad ②$$

substitute 2 in the μ equation

$$\mu = \frac{E}{2\left(1+\frac{\lambda}{2(\mu+\lambda)}\right)}$$

$$\mu = \frac{E}{\cancel{2}\left(\frac{2\mu+3\lambda}{\cancel{2}(\mu+\lambda)}\right)}$$

$$\boxed{E = \mu\left(\frac{2\mu+3\lambda}{\mu+\lambda}\right)}$$

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b) substituting the previously obtained relations in the plane stress elastic matrix

$$\underline{\underline{C}} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} = \frac{\mu(2\mu+3\lambda)}{(\mu+\lambda)\left(1-\frac{\lambda^2}{4(\mu+\lambda)^2}\right)}\begin{bmatrix} 1 & \frac{\lambda}{2(\mu+\lambda)} & 0 \\ \frac{\lambda}{2(\mu+\lambda)} & 1 & 0 \\ 0 & 0 & \frac{2\mu+\lambda}{4(\mu+\lambda)} \end{bmatrix}$$

$$= \frac{\mu\cancel{(2\mu+3\lambda)}}{(\mu+\lambda)\left(\frac{2\mu+\lambda}{2(\mu+\lambda)}\right)\left(\frac{\cancel{2\mu+3\lambda}}{2(\mu+\lambda)}\right)}\begin{bmatrix} 1 & \frac{\lambda}{2(\mu+\lambda)} & 0 \\ \frac{\lambda}{2(\mu+\lambda)} & 1 & 0 \\ 0 & 0 & \frac{2\mu+\lambda}{4(\mu+\lambda)} \end{bmatrix}$$

$$= \frac{4\mu(\mu+\lambda)}{2\mu+\lambda}\begin{bmatrix} 1 & \frac{\lambda}{2(\mu+\lambda)} & 0 \\ \frac{\lambda}{2(\mu+\lambda)} & 1 & 0 \\ 0 & 0 & \frac{2\mu+\lambda}{4(\mu+\lambda)} \end{bmatrix}$$

<u>For plane strain</u>

$$\underline{\underline{C}} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} = \frac{\mu\cancel{(2\mu+3\lambda)}}{\cancel{(\mu+\lambda)}\left(\frac{\cancel{2\mu+3\lambda}}{2(\mu+\lambda)}\right)\left(\frac{\cancel{\mu}}{\cancel{\mu+\lambda}}\right)}\begin{bmatrix} \frac{2\mu+\lambda}{2(\mu+\lambda)} & \frac{\lambda}{2(\mu+\lambda)} & 0 \\ \frac{\lambda}{2(\mu+\lambda)} & \frac{2\mu+\lambda}{2(\mu+\lambda)} & 0 \\ 0 & 0 & \frac{\mu}{2(\mu+\lambda)} \end{bmatrix}$$

$$= 2(\mu+\lambda)\begin{bmatrix} \frac{2\mu+\lambda}{2(\mu+\lambda)} & \frac{\lambda}{2(\mu+\lambda)} & 0 \\ \frac{\lambda}{2(\mu+\lambda)} & \frac{2\mu+\lambda}{2(\mu+\lambda)} & 0 \\ 0 & 0 & \frac{\mu}{2(\mu+\lambda)} \end{bmatrix} = \begin{bmatrix} 2\mu+\lambda & \lambda & 0 \\ \lambda & 2\mu+\lambda & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

c) The stress-strain matrix for plane strain:

$$E = \begin{bmatrix} 2\mu+\lambda & \lambda & 0 \\ \lambda & 2\mu+\lambda & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

splitting the matrix into E_μ , E_λ

$$E_\mu = \begin{bmatrix} 2\mu & 0 & 0 \\ 0 & 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

$$E_\lambda = \begin{bmatrix} \lambda & \lambda & 0 \\ \lambda & \lambda & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

in which $E = E_\mu + E_\lambda$

D) E_μ and E_λ in terms of E and ν

$$E_\mu = \begin{bmatrix} \frac{E}{1+\nu} & 0 & 0 \\ 0 & \frac{E}{1+\nu} & 0 \\ 0 & 0 & \frac{E}{2(1+\nu)} \end{bmatrix} = \frac{E}{1+\nu} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$E_\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Assignment 3.3

1 The Stiffness matrices

1.1 For plane stress triangle with $\nu = 0$:

$$t = 1, \qquad A = 0.5a^2 = 0.5$$

$$\mathbf{K} = \mathbf{B^T E B} tA = 0.5E \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$= 0.25E \begin{bmatrix} 3 & 1 & -2 & -1 & -1 & 0 \\ 1 & 3 & 0 & -1 & -1 & -2 \\ -2 & 0 & 2 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \end{bmatrix}$$

1.2 For the 3-bar element:

1-2 bar with angle zero:

$$K_{12} = EA_2 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The expanded form

$$K_{12} = EA_2 \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

1-2 bar with angle 90:

$$K_{12} = EA_1 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix}$$

The expanded form

$$K_{12} = EA_1 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

2-3 bar with angle 135:

$$K_{12} = 0.7071EA_3 \begin{bmatrix} 0.5000 & -0.5000 & -0.5000 & 0.5000 \\ -0.5000 & 0.5000 & 0.5000 & -0.5000 \\ -0.5000 & 0.5000 & 0.5000 & -0.5000 \\ 0.5000 & -0.5000 & -0.5000 & 0.5000 \end{bmatrix}$$

The expanded form

$$K_{12} = 0.7071 E A_3 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5000 & -0.5000 & -0.5000 & 0.5000 \\ 0 & 0 & -0.5000 & 0.5000 & 0.5000 & -0.5000 \\ 0 & 0 & -0.5000 & 0.5000 & 0.5000 & -0.5000 \\ 0 & 0 & 0.5000 & -0.5000 & -0.5000 & 0.5000 \end{bmatrix}$$

The assembled K matrix, using $A_1 = A_2 = A$ and $A_3 = A'$

$$K = K_{12} + K_{13} + K_{23} = E \begin{bmatrix} A & 0 & -A & 0 & 0 & 0 \\ 0 & A & 0 & 0 & 0 & -A \\ -A & 0 & A + \frac{\sqrt{2}A'}{4} & -\frac{\sqrt{2}A'}{4} & -\frac{\sqrt{2}A'}{4} & \frac{\sqrt{2}A'}{4} \\ 0 & 0 & -\frac{\sqrt{2}A'}{4} & \frac{\sqrt{2}A'}{4} & \frac{\sqrt{2}A'}{4} & -\frac{\sqrt{2}A'}{4} \\ 0 & 0 & -\frac{\sqrt{2}A'}{4} & \frac{\sqrt{2}A'}{4} & \frac{\sqrt{2}A'}{4} & -\frac{\sqrt{2}A'}{4} \\ 0 & -A & \frac{\sqrt{2}A'}{4} & -\frac{\sqrt{2}A'}{4} & -\frac{\sqrt{2}A'}{4} & A + \frac{\sqrt{2}A'}{4} \end{bmatrix}$$

2 Values to make the matrices similar

The 2 matrices can never be equivalent, however we can choose values of A and A' to make them close as possible. This is done by equating the diagonal terms.

The values that lead to having most of the diagonal terms to be equal are

$$A = \frac{3}{4}, \quad A' = 0.7071$$

Another way is to take an average for A and A' values leading to

$$A = 0.5883, \quad A' = 0.1125$$

3 Reason for not being equal

The two stiffness matrices are not equal because they represent two different structures. The first one represents a solid triangle with thickness of 1 that is represented by 2d triangle element in FEA. The second one represents 3 connected bars, which are represented by 1d bar elements.

4 Plane stress with ν not equal zero

doing the same procedure in a) but with a different Elastic matrix will lead to:

$$K = 0.25 \frac{E}{1-\nu^2} \begin{bmatrix} 3-\nu & \nu+1 & -2 & \nu-1 & \nu-1 & -2\nu \\ \nu+1 & 3-\nu & -2\nu & \nu-1 & \nu-1 & -2 \\ -2 & -2\nu & 2 & 0 & 0 & 2\nu \\ \nu-1 & \nu-1 & 0 & 1-\nu & 1-\nu & 0 \\ \nu-1 & \nu-1 & 0 & 1-\nu & 1-\nu & 0 \\ -2\nu & -2 & 2\nu & 0 & 0 & 2 \end{bmatrix}$$

This stiffness matrix is different from the first one in the consideration of the transverse strain that plays a role in the induced stresses which is the role of Poisson's ratio. We can notice that setting $\nu = 0$ will lead to a).

The K matrix of the bar is not changed since ν doesn't play any role.