Computational Structural Mechanics and Dynamics

Master of Science in Computational Mechanics
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Homework 3

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Assignment 3.1

On "The Plane Stress Problem":
In isotropic elastic materials (as well as in plasticity and viscoelasticity) it is convenient to use the so-called Lamé constants λ and μ instead of E and ν in the constitutive equations. Both λ and μ have the physical dimension of stress and are related to E and ν by

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \qquad \mu = G = \frac{E}{2(1+\nu)}$$

1. Find the inverse relations for E, ν in terms of λ, μ.

$$\nu = \frac{\lambda}{2(\mu+\lambda)} \qquad E = \frac{\mu(3\mu+2\lambda)}{\mu+\lambda}$$

2. Express the elastic matrix for plane stress and plane strain cases in terms of λ, μ.
<u>Elastic matrix for plane stress:</u>

$$\frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu/2 \end{bmatrix} \rightarrow \frac{8\mu^3+20\lambda\mu^2+12\lambda^2\mu}{4\mu^2+3\lambda^2}\begin{bmatrix} 1 & \lambda/2(\mu+\lambda) & 0 \\ \lambda/2(\mu+\lambda) & 1 & 0 \\ 0 & 0 & 2\mu+\lambda/4(\mu+\lambda) \end{bmatrix}$$

<u>Elastic matrix for plane strain:</u>

$$\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1 & \nu/(1-\nu) & 0 \\ \nu/(1-\nu) & 1 & 0 \\ 0 & 0 & (1-2\nu)/2(1-\nu) \end{bmatrix} \rightarrow \begin{bmatrix} 2\mu+\lambda & \lambda & 0 \\ \lambda & 2\mu+\lambda & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

3. Split the stress-strain matrix E for plane strain as

$$\mathbf{E} = \mathbf{E}_\lambda + \mathbf{E}_\mu$$

in which E_μ and E_λ contain only μ and λ, respectively.

This is the Lamé λ,μ splitting of the plane strain constitutive equations, which leads to the so-called B-bar formulation of near-incompressible finite elements.

$$\mathbf{E} = \begin{bmatrix} \lambda & \lambda & 0 \\ \lambda & \lambda & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 2\mu & 0 & 0 \\ 0 & 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

4. Express E_λ and E_μ also in terms of E and ν.

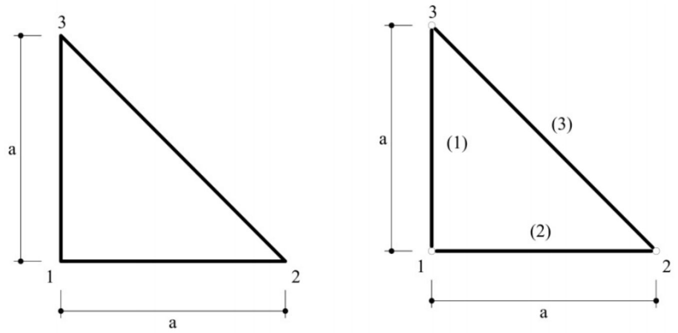
$$\mathbf{E} = \frac{E\nu}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{E}{2(1+\nu)}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Assignment 3.2

On "The 3-node Plane Stress Triangle":
Consider a plane triangular domain of thickness h, with horizontal and vertical edges of length a. Let us consider for simplicity $a = 1$, $h = 1$. The material parameters are E, ν. Initially ν is set to zero. Two discrete structural models are considered as depicted in the figure:

(a) A plane linear Turner triangle with the same dimensions.

(b) A set of three bar elements placed over the edges of the triangular domain. The cross sections for the bars are $A_1 = A_2$ and A_3.



1. Calculate the stiffness matrices K_{tri} and K_{bar} for both discrete models.

$$K_{tri} = \int_{\Omega^e} h\mathbf{B^t E B}\, d\Omega \qquad K_{tri} = h\mathbf{B^t E B}\int_{\Omega^e} d\Omega$$

Defining $\mathbf{B}$ as:

$$\begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} \end{bmatrix} \qquad J = J^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{bmatrix} = [J^{-1}] \begin{bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{bmatrix}$$

With $N_1 = 1 - \xi - \eta$, $N_2 = \xi$, $N_3 = \eta$

$$= \frac{h}{2} \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} E & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & \frac{E}{2} \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$K_{tri} = \begin{bmatrix} 3E/4 & E/4 & -E/2 & -E/4 & -E/4 & 0 \\ E/4 & 3E/4 & 0 & -E/4 & -E/4 & -E/2 \\ -E/2 & 0 & E/2 & 0 & 0 & 0 \\ -E/4 & -E/4 & 0 & E/4 & E/4 & 0 \\ -E/4 & -E/4 & 0 & E/4 & E/4 & 0 \\ 0 & -E/2 & 0 & 0 & 0 & E/2 \end{bmatrix}$$

For K_{tri}, taking accout different bar elements with $A_1 = A_2$ and A_3

$$K^{(1)} = E \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A & 0 & 0 & 0 & -A \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -A & 0 & 0 & 0 & A \end{bmatrix} \quad K^{(2)} = E \begin{bmatrix} A & 0 & -A & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -A & 0 & A & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K^{(3)} = E \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_3/2\sqrt{2} & -A_3/2\sqrt{2} & -A_3/2\sqrt{2} & A_3/2\sqrt{2} \\ 0 & 0 & -A_3/2\sqrt{2} & A_3/2\sqrt{2} & A_3/2\sqrt{2} & -A_3/2\sqrt{2} \\ 0 & 0 & -A_3/2\sqrt{2} & A_3/2\sqrt{2} & A_3/2\sqrt{2} & -A_3/2\sqrt{2} \\ 0 & 0 & A_3/2\sqrt{2} & -A_3/2\sqrt{2} & -A_3/2\sqrt{2} & A_3/2\sqrt{2} \end{bmatrix}$$

$$K_{bar} = E \begin{bmatrix} A & 0 & -A & 0 & 0 & 0 \\ 0 & A & 0 & 0 & 0 & -A \\ -A & 0 & A + A_3/2\sqrt{2} & -A_3/2\sqrt{2} & -A_3/2\sqrt{2} & A_3/2\sqrt{2} \\ 0 & 0 & -A_3/2\sqrt{2} & A_3/2\sqrt{2} & A_3/2\sqrt{2} & -A_3/2\sqrt{2} \\ 0 & 0 & -A_3/2\sqrt{2} & A_3/2\sqrt{2} & A_3/2\sqrt{2} & -A_3/2\sqrt{2} \\ 0 & -A & A_3/2\sqrt{2} & -A_3/2\sqrt{2} & -A_3/2\sqrt{2} & A + A_3/2\sqrt{2} \end{bmatrix}$$

2. Is there any set of values for the cross sections $A_1 = A_2$ and A_3 to make both stiffness matrix equivalent: $K_{bar} = K_{tri}$ If not, which are the values that make them more similar?

   The values $A_3 = -E\sqrt{2}$ and $A = 3E/4$ are the ones that make the stiffness matrices to be more similar.

3. Why these two stiffness matrices are not equal? Find a physical explanation.

   Both matrices are different, due to the location of the information for each element (bars and solid triangle).

   In terms of information, both matrix contains 14 values with no data inside the matrix. The difference lives on their location. Meanwhile, the solid triangle element, has distributed non-zero values, on the truss structure that zeros values are concentrate on the interior of the triangular shape.

   In the following question, taking ν different from zero, we can appreciate how the stiffness value increase for K_{tri}, arising the maximun information for computing an accurate solution.

4. Considering nowidering $\nu \neq 0$ and extract some conclusions.

   With value ν different of zero, we recover more information for K_{tri}. The stiffness matrix is recovering 8 values, deffining higher accuracy on the solution.