Universitat Polytechnica de Catalunya
MSc Computational Mechanics
Spring 2018

Computational Structural Mechanics and Dynamics

Assignment 3

Due 26/02/2018

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1 Assignment 3.1.1

For the first part of this assignment, we will Find the inverse relations for E, ν in terms of λ, μ. This can be seen in the figure below.

1) Find the inverse relations for E, ν in terms of λ, μ

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \qquad \mu = \frac{E}{2(1+\nu)} \qquad (1)$$

– first we will rearrange μ for E in (1)

$$E = 2\mu(1+\nu) \qquad (2)$$

– we will now plug in E into λ from (1)

$$\lambda = \frac{2\mu(1+\nu)\nu}{(1+\nu)(1-2\nu)} \qquad (3)$$

– reducing gives us...

$$\lambda = \frac{2\mu\nu}{(1-2\nu)} \qquad (4)$$

– rearranging for ν...

$$\lambda - 2\nu\lambda = 2\mu\nu$$

$$\lambda = 2\mu\nu + 2\nu\lambda$$

$$\lambda = 2(\mu+\lambda)\nu$$

$$\boxed{\nu = \frac{\lambda}{2(\mu+\lambda)}} \qquad (5)$$

– we will now plug (5) into (2)

$$E = 2\mu\left(1 + \frac{\lambda}{2(\mu+\lambda)}\right) = 2\mu\left(\frac{2\mu+3\lambda}{2(\mu+\lambda)}\right)$$

$$\boxed{E = \mu\left(\frac{2\mu+3\lambda}{\mu+\lambda}\right)} \qquad (6)$$

2 Assignment 3.1.2

We will now transform the plane stress and plane strain elasticity matrices in terms of λ, μ.

2) The plane stress elasticity matrix is ...

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \end{bmatrix} \qquad (7)$$

– we will now solve for $\frac{E}{1-\nu^2}$, $\frac{1-\nu}{2}$ in terms of λ, μ by plugging in (5), (6)

$$\frac{E}{1-\nu^2} = \frac{\left[\mu\left(\frac{2\mu+3\lambda}{\mu+\lambda}\right)\right]}{\left[1 - \frac{\lambda^2}{4(\lambda+\mu)^2}\right]} = \frac{\left[\mu\left(\frac{2\mu+3\lambda}{\mu+\lambda}\right)\right]}{\left[\frac{4(\mu+\lambda)^2+\lambda^2}{4(\mu+\lambda)^2}\right]} \qquad (8)$$

– reducing (8) gives us..

$$\frac{E}{1-\nu^2} = \frac{4\mu(2\mu+3\lambda)}{\left[\frac{(2(\mu+\lambda)+\lambda)(2(\mu+\lambda)-\lambda)}{(\mu+\lambda)}\right]} = \frac{4\mu(\mu+\lambda)}{2\mu+\lambda} \qquad (9)$$

– now solving $\frac{1-\nu}{2}$ in terms of λ, μ

$$\frac{1-\nu}{2} = \frac{\left[1 - \frac{\lambda}{2(\mu+\lambda)}\right]}{2} = \frac{\left[\frac{2\mu+\lambda}{2(\mu+\lambda)}\right]}{2} = \frac{2\mu+\lambda}{4(\mu+\lambda)} \qquad (10)$$

– and plugging (10), (9), (5) into (7) gives us

$$\boxed{\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{bmatrix} = \frac{4\mu(\mu+\lambda)}{2\mu+\lambda} \begin{bmatrix} 1 & \frac{\lambda}{2(\mu+\lambda)} & 0 \\ \frac{\lambda}{2(\mu+\lambda)} & 1 & 0 \\ 0 & 0 & \frac{2\mu+\lambda}{4(\mu+\lambda)} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \end{bmatrix}}$$

The plane strain problem elasticity matrix is ...

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{bmatrix} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & 0 \\ \frac{\nu}{1-\nu} & 1 & 0 \\ 0 & 0 & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{zz} \end{bmatrix} \qquad (11)$$

- we will now solve for $\frac{\nu}{1-\nu}, \frac{1-2\nu}{2(1-\nu)}, \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$ in terms of λ, μ by plugging in (5), (6)

$$\frac{\nu}{1-\nu} = \frac{\left(\frac{\lambda}{2(\mu+\lambda)}\right)}{\left(\frac{2\mu+\lambda}{2(\mu+\lambda)}\right)} = \frac{\lambda}{2\mu+\lambda} \qquad (12)$$

$$\frac{1-2\nu}{2(1-\nu)} = \frac{\left(1-\left(\frac{2\lambda}{2(\mu+\lambda)}\right)\right)}{\left(2\left(\frac{2\mu+\lambda}{2(\mu+\lambda)}\right)\right)} = \frac{\left(\frac{\mu}{\mu+\lambda}\right)}{\left(\frac{2\mu+\lambda}{\mu+\lambda}\right)} = \frac{\mu}{2\mu+\lambda} \qquad (13)$$

$$\frac{E(1-\nu)}{(1+\nu)(1-2\nu)} = \frac{\left[\frac{\mu(2\mu+3\lambda)(2\mu+\lambda)}{2(\mu+\lambda)(\mu+\lambda)}\right]}{\left[\frac{\mu(2\mu+3\lambda)}{2(\mu+\lambda)(\mu+\lambda)}\right]} = 2\mu+\lambda \qquad (14)$$

- now, plugging (12),(13),(14) back into (11) gives us

$$\boxed{\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{bmatrix} = (2\mu+\lambda) \begin{bmatrix} 1 & \frac{\lambda}{2\mu+\lambda} & 0 \\ \frac{\lambda}{2\mu+\lambda} & 1 & 0 \\ 0 & 0 & \frac{\mu}{2\mu+\lambda} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{zz} \end{bmatrix}} \qquad (15)$$

3 Assignment 3.1.3

Now we will split the plane strain matrix into its components as follows.

3) we will now split the stress-strain matrix E for plain strain as $E = E_\lambda + E_\mu$

- first we will distribute the terms from (15) this gives us

$$\begin{bmatrix} 2\mu+\lambda & \frac{\lambda(2\mu+\lambda)}{2\mu+\lambda} & 0 \\ \frac{\lambda(2\mu+\lambda)}{2\mu+\lambda} & 2\mu+\lambda & 0 \\ 0 & 0 & \frac{\mu(2\mu+\lambda)}{(2\mu+\lambda)} \end{bmatrix} \tag{16}$$

reducing (16) gives us

$$\begin{bmatrix} 2\mu+\lambda & \lambda & 0 \\ \lambda & 2\mu+\lambda & 0 \\ 0 & 0 & \mu \end{bmatrix} \tag{17}$$

and splitting (17) gives us

$$\boxed{E_\lambda = \lambda \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad E_\mu = \mu \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

4 Assignment 3.1.4

And replacing λ, μ with their representations in terms of E and ν give us our final answer.

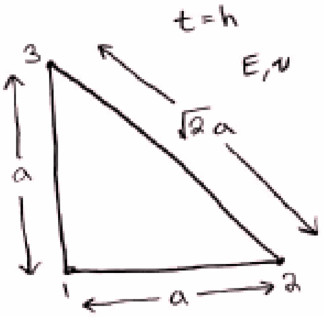
$$E_\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$E_\mu = \frac{E}{2(1+\nu)} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

5 Assignment 3.2.1

We will now calculate both K_{tri} and K_{bar} so we will be able to compare them in later sections. These calculations can be seen below and on the next page.

a) plane linear turner triangle with same dimensions



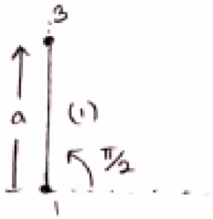
$$y_{12} = 0 \qquad x_{32} = -a$$
$$y_{23} = -a \qquad x_{13} = 0 \qquad A = \tfrac{1}{2}a^2$$
$$y_{31} = a \qquad x_{21} = a$$

$$K_{tri} = \frac{E}{2a^2}\begin{bmatrix} -a & 0 & -a \\ 0 & -a & -a \\ a & 0 & 0 \\ 0 & 0 & a \\ 0 & 0 & a \\ 0 & a & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} -a & 0 & a & 0 & 0 & 0 \\ 0 & -a & 0 & 0 & 0 & a \\ -a & -a & 0 & a & a & 0 \end{bmatrix}$$

$$\boxed{K_{tri} = \frac{E}{2}\begin{bmatrix} 1.5 & 0.5 & -1.0 & -0.5 & -0.5 & 0 \\ 0.5 & 1.5 & 0 & -0.5 & -0.5 & -1.0 \\ -1.0 & 0 & 1.0 & 0 & 0 & 0 \\ -0.5 & -0.5 & 0 & 0.5 & 0.5 & 0 \\ -0.5 & -0.5 & 0 & 0.5 & 0.5 & 0 \\ 0 & -1.0 & 0 & 0 & 0 & 1.0 \end{bmatrix}}$$

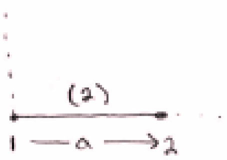
b) set of 3 bar elements $A_1 = A_2$, A_3

Element 1



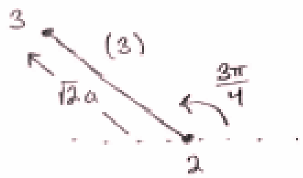
$$K^1 = EA_1 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

Element 2



$$K^2 = EA_1 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Element 3



$$K^3 = \frac{EA_3}{\sqrt{2}} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$K_{BAR} = E \begin{bmatrix} A & 0 & -A & 0 & 0 & 0 \\ 0 & A & 0 & 0 & 0 & -A \\ -A & 0 & A_1 + \frac{A_3}{2\sqrt{2}} & \frac{-A_3}{2\sqrt{2}} & \frac{-A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} \\ 0 & 0 & \frac{-A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} & \frac{-A_3}{2\sqrt{2}} \\ 0 & 0 & \frac{-A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} & \frac{-A_3}{2\sqrt{2}} \\ 0 & -A_1 & \frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} + A_1 \end{bmatrix}$$

6 Assignment 3.2.2

There are no values of A_3 to make the stiffness matrices equal. A value that will make some of the terms similar is $A_3 = 1/\sqrt{2}$. This will bring some of the terms in the same position to $1/4$.

7 Assignment 3.2.3

The physical meaning behind these matrices not being equal is that when the element is made up of 3 bar elements, it acts as more of a truss structure with no material inside the element. On the other hand, the triangular element acts more like a cross section of area with material inside the element.

8 Assignment 3.2.4

Changing Possion's ratio to a nonzero value will change the elastic matrix. This can be seen below.

when $\nu \neq 0$, the elastic strain matrix is as follows

$$\bar{E} = \frac{E}{1-\nu}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

It is obvious to see that changing ν will have a big impact on the final K_{tri} stiffness matrix

This really helps highlight the physical differences between the two types of element. The bar elements are not even dependent on Possion's ratio adding strength to the theory that it behaves more like a truss and not like a cross-section of material.