UNIVERSITAT POLITÈCNICA DE CATALUNYA

COMPUTATIONAL SOLID MECHANICS AND DYNAMICS

MASTER'S DEGREE IN NUMERICAL METHODS IN ENGINEERING

On 'Plane stress problem' and 'Linear Traingle'

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1 Assignment 3.1

1.1 a)

The choice of triangular elements for the analysis of problems involving two dimensions is suitable for this geometry and other arbitrarily shaped domains as it allows a close approximation to their shape. The displacement field on the triangular nodes is defined in terms of their two components u and v in the x and y directions, respectively, with the same shape function to represent both displacements.

If writing a relation between the coordinates of the nodes of an element and the Cartesian coordinates, it is obtained that

$$\begin{cases} x = N_i x_i + N_j x_j + N_k x_k \\ y = N_i y_i + N_j y_j + N_k y_k \end{cases} \tag{1}$$

Therefore the shape functions are chosen such that N_i must be a function which is unity at vertex i. Specifically,

$$N_i = \frac{a_i + b_i x - c_i y}{2A^e} \tag{2}$$

With A^e being the area of the element and

$$\begin{cases} a_i = x_j y_k - x_k y_j \\ b_i = y_j - y_k \\ c_i = x_k - x_j \end{cases} \tag{3}$$

Now that the shape functions have been stated cleary, it is time to go further on expressing the weighted residual statement for the equilibrium in terms of the stresses. The stresses are defined as

$$\sigma = \frac{E}{1-v^2} \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix} \epsilon = \mathbf{D}\epsilon \tag{4}$$

$\mathbf{K^e_{l,m}}$ will be (particularized for the triangular element)

$$\mathbf{K^e_{l,m}} = \left(\frac{t^e}{4A^e}\right) \begin{bmatrix} b_i b_j d_{11} + c_i c_j d_{33} & b_i c_j d_{12} + b_j c_i d_{33} \\ c_i b_j d_{21} + b_i c_j d_{33} & b_i b_j d_{33} + c_i c_j d_{22} \end{bmatrix} \tag{5}$$

It is important to note that this matrix will always be symmetric as $d_{12} = d_{21}$. Now, the whole domain is depicted in Fig. 1, with each node and element numbered.

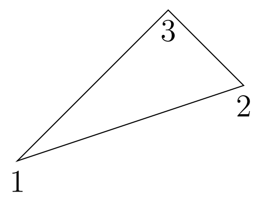

Figure 1: Element descriptionn.

The coordinates of the triangle are computed below in a matrix $\mathbf{X}$ that will be later used in the Matlab code the matrix.

$$\mathbf{X} = \begin{bmatrix} \mathbf{x_i} \\ \\ \mathbf{y_i} \end{bmatrix} \quad (i = 1, 2, 3) \quad = \begin{bmatrix} 0 & 3 & 2 \\ \\ 0 & 1 & 2 \end{bmatrix}^T \tag{6}$$

Then, the connectivity matrix is simply constructed by allocating in each row the global numbering of the nodes that form each element. The connectivity matrix is straightforward.

$$\mathbf{T} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \tag{7}$$

For the triangle element,

$$\begin{cases} b_1 = \mathbf{X}(2,2) - \mathbf{X}(3,2) \\ c_1 = \mathbf{X}(3,1) - \mathbf{X}(2,1) \\ b_2 = \mathbf{X}(3,2) - \mathbf{X}(1,2) \\ c_2 = \mathbf{X}(1,1) - \mathbf{X}(3,1) \\ b_3 = \mathbf{X}(1,2) - \mathbf{X}(2,2) \\ c_3 = \mathbf{X}(2,1) - \mathbf{X}(1,1) \end{cases} \tag{8}$$

Now it is possible to construct the stiffness matrices for the element. In doing so, it is obtained

$$
\mathbf{K} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
\begin{array}{c}
\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\begin{pmatrix}
K_{11}^{(1)} & K_{12}^{(1)} & K_{13}^{(1)} & K_{14}^{(1)} & K_{15}^{(1)} & K_{16}^{(1)} \\
 & K_{22}^{(1)} & K_{23}^{(3)} & K_{24}^{(1)} & K_{24}^{(1)} & K_{26}^{(1)} \\
 & & K_{33}^{(1)} & K_{34}^{(1)} & K_{35}^{(1)} & K_{36}^{(1)} \\
 & \mathit{Symm.} & & K_{44}^{(1)} & K_{45}^{(1)} & K_{46}^{(1)} \\
 & & & & K_{55}^{(1)} & K_{56}^{(1)} \\
 & & & & & K_{66}^{(1)}
\end{pmatrix}
\end{array}
$$

Given the area of the element (2) and its thickness (1), the matrix is obtained rapidly with Matlab. The code may be seen in the Annex and it has been adapted from another code developed in the subject "Finite Element Method"

$$
K = \begin{bmatrix}
18.75 & 9.375 & -12.50 & -6.25 & -6.25 & -3.125 \\
9.375 & 18.75 & 6.25 & 12.50 & -15.625 & -31.25 \\
-12.50 & 6.25 & 75 & -37.50 & -62.50 & 31.25 \\
-6.25 & 12.50 & -37.50 & 75 & 43.75 & -87.50 \\
-6.25 & -15.625 & -62.50 & 43.75 & 68.75 & -28.125 \\
-3.125 & -31.25 & 31.25 & -87.50 & -28.125 & 118.75
\end{bmatrix} \tag{9}
$$

1.2 b)

Because of the properties of the shape functions, the sum of the entries in each row and column is zero. This can be easily checked with Matlab or a calculator once the matrix is obtained. The reason is because the linear shape function takes values of unity on the current node and zero at the others.

2 Assignment 3.2

2.1 a)

Returning to equations (4) and (5), the matrix for the linear triangle is obtained straighforwar considering that υ is zero and both the length and the thickness are 1.

$$\sigma = \frac{E}{1-v^2}\begin{bmatrix}1 & v & 0\\ v & 1 & 0\\ 0 & 0 & \frac{1-v}{2}\end{bmatrix} = E\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \frac{1}{2}\end{bmatrix}\epsilon = \mathbf{D}\epsilon \tag{10}$$

$\mathbf{K^e_{l,m}}$ will be (particularized for the triangular element)

$$\mathbf{K^e_{l,m}} = \left(\frac{t^e}{4A^e}\right)\begin{bmatrix}b_ib_jd_{11}+c_ic_jd_{33} & b_ic_jd_{12}+b_jc_id_{33}\\ c_ib_jd_{21}+b_ic_jd_{33} & b_ib_jd_{33}+c_ic_jd_{22}\end{bmatrix} = \left(\frac{1}{2}\right)\begin{bmatrix}b_ib_j+c_ic_j/2 & b_jc_i/2\\ b_ic_j/2 & b_ib_j/2\end{bmatrix} \tag{11}$$

Introducing this data into the Matlab code, the stiffness matrix is simply

$$K = E\begin{bmatrix}0.75 & 0.25 & -0.50 & -0.25 & -0.25 & 0\\ 0.25 & 0.75 & 0 & -0.25 & -0.25 & -0.5\\ -0.5 & 0 & 0.50 & 0 & 0 & 0\\ -0.25 & -0.25 & 0 & 0.25 & 0.25 & -0\\ -0.25 & -0.25 & 0 & 0.25 & 0.25 & -0\\ 0 & -0.5 & 0 & 0 & 0 & 0.5\end{bmatrix} \tag{12}$$

We can clearly see that this matrix is singular.

2.2 b)

As for the elemental bars structure, it can be seen from previous assignments that the individual stiffness matrices are

$$K^{(1)}_{11} = K^{(1)}_{22} = -K^{(1)}_{12} = -K^{(1)}_{21} = (EA_1)\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} \tag{13}$$

$$K^{(2)}_{11} = K^{(2)}_{22} = -K^{(2)}_{12} = -K^{(2)}_{21} = (EA_2)\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} \tag{14}$$

$$K_{11}^{(3)} = K_{22}^{(3)} = -K_{12}^{(3)} = -K_{21}^{(3)} = \frac{EA_3}{\sqrt{2}} \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix} \tag{15}$$

$$K = E \begin{bmatrix} A_2 & 0 & -A_2 & 0 & 0 & 0 \\ 0 & A_1 & 0 & 0 & 0 & -A_1 \\ -A_2 & 0 & A_2 + \frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} \\ 0 & 0 & -\frac{A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} \\ 0 & 0 & -\frac{A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} \\ 0 & -A_1 & \frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} & A_1 + \frac{A_3}{2\sqrt{2}} \end{bmatrix} \tag{16}$$

Now if substituting $A_1 = A_2 = A, A_3 = A'$ the matrix is

$$K = E \begin{bmatrix} A & 0 & -A & 0 & 0 & 0 \\ 0 & A & 0 & 0 & 0 & -A \\ -A & 0 & A + \frac{A'}{2\sqrt{2}} & -\frac{A'}{2\sqrt{2}} & -\frac{A'}{2\sqrt{2}} & \frac{A'}{2\sqrt{2}} \\ 0 & 0 & -\frac{A'}{2\sqrt{2}} & \frac{A'}{2\sqrt{2}} & \frac{A'}{2\sqrt{2}} & -\frac{A'}{2\sqrt{2}} \\ 0 & 0 & -\frac{A'}{2\sqrt{2}} & \frac{A'}{2\sqrt{2}} & \frac{A'}{2\sqrt{2}} & -\frac{A'}{2\sqrt{2}} \\ 0 & -A & \frac{A'}{2\sqrt{2}} & -\frac{A'}{2\sqrt{2}} & -\frac{A'}{2\sqrt{2}} & A + \frac{A'}{2\sqrt{2}} \end{bmatrix} \tag{17}$$

2.3 c)

This matrix cannot be made equivalent to the one of the Turner triangle as there are are more zero entries in the bar structure, which is to be expected. If we were to put $A = 1/2, A' = 1/\sqrt{2}$, then we would obtain several values in the diagonal term to be equal, although not all of them. Obviously as the two matrices represent the global stiffness of different structural problems (one of bars and another of a triangle structure as a whole) the problem cannot be made equivalent. The bar triangle is constructed with hinges, which means that the elements can only hold axial stresses and boundary conditions applied a the nodes only, whereas the Turner triangle can be applied distributed loads along its perimeter. Therefore there will be more zero elements in this matrix as in the Turner triangle, in which a prescribed displacement in any point will affect the whole structure, whereas the same displacement in the bar structure may produce little effect.

2.4 d)

If considering again equation (4), it can be seen that the Poisson ratio relates the strains in the different coordinates to the stresses. That is, if the Poisson ratio is zero, the stresses are given by a diagonal matrix, meaning that only a displacement in the direction of the axis will cause stresses in that axis. If the Poisson ratio is non-zero, the whole structure will be more stiff as stresses will appear even if the strains are not in the same direction. Incorporating now the Poisson ratio, the matrix is given by Matlab such that

$$K = \frac{E}{1-\upsilon^2}\begin{bmatrix} 0.75-\frac{\upsilon}{4} & \frac{\upsilon}{4}+0.25 & -0.50 & \frac{\upsilon}{4}-0.25 & \frac{\upsilon}{4}-0.25 & -\frac{\upsilon}{2} \\ \frac{\upsilon}{4}+0.25 & 0.75-\frac{\upsilon}{4} & -\frac{\upsilon}{2} & \frac{\upsilon}{4}-0.25 & \frac{\upsilon}{4}-0.25 & -0.5 \\ -0.5 & -\frac{\upsilon}{2} & 0.50 & 0 & 0 & \frac{\upsilon}{2} \\ \frac{\upsilon}{4}-0.25 & \frac{\upsilon}{4}-0.25 & 0 & 0.25-\frac{\upsilon}{4} & 0.25-\frac{\upsilon}{4} & -0 \\ \frac{\upsilon}{4}-0.25 & \frac{\upsilon}{4}-0.25 & 0 & 0.25-\frac{\upsilon}{4} & 0.25-\frac{\upsilon}{4} & -0 \\ -\frac{\upsilon}{2} & -0.5 & \frac{\upsilon}{2} & 0 & 0 & 0.5 \end{bmatrix} \tag{18}$$

A Appendix:Code

```
format long
clear all

E = [100 25 0; 25 100 0; 0 0 50];

C = [0 0; 3 1; 2 2];

T = [1 2 3];

for i = 1:1
   [Area(i),A{i}] = elem_area(C,T(i,:));
end

for i = 1:3
    for j = 1:3
        K{i,j} = matrix_K(i,j,Area(1),A{1},E);
    end
end

Glob = [K{1,1} K{1,2} K{1,3}; ...
```

```
    K{2,1} K{2,2} K{2,3};...
    K{3,1} K{3,2} K{3,3}];
```

```
function [Area,A] = elem_area(C,T)
M = [1 C(T(1),1) C(T(1),2); 1 C(T(2),1) C(T(2),2);1 C(T(3),1) C(T(3),2)];
Area = 0.5*det(M);

b1 = C(T(2),2) - C(T(3),2);
c1 = C(T(3),1) - C(T(2),1);

b2 = C(T(3),2) - C(T(1),2);
c2 = C(T(1),1) - C(T(3),1);

b3 = C(T(1),2) - C(T(2),2);
c3 = C(T(2),1) - C(T(1),1);

A = [0 0 0; b1 b2 b3; c1 c2 c3];
end
```

```
function K = matrix_K(i,j,Area,A,E)

K = [A(2,i)*A(2,j)*E(1,1) + A(3,i)*A(3,j)*E(3,3) ...
    A(2,i)*A(3,j)*E(1,2) + A(2,j)*A(3,i)*E(3,3); ...
    A(3,i)*A(2,j)*E(2,1) + A(2,i)*A(3,j)*E(3,3) ...
    A(2,i)*A(2,j)*E(3,3) + A(3,i)*A(3,j)*E(2,2)];
K = K/(4*Area);
```