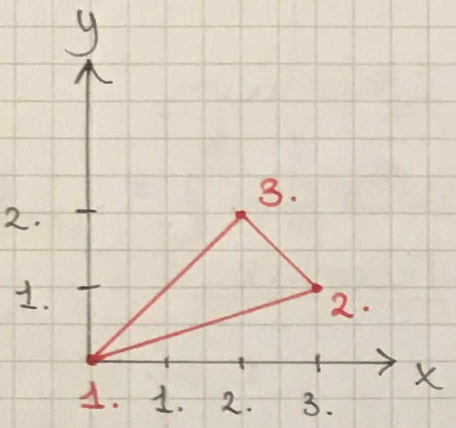
Assigment 3 - Computational Structural Mechanics and Dynamichs

Assignment 3.1

1) Plane stress triangle $\Rightarrow$ $x_1 = 0$, $y_1 = 0$

$x_2 = 3$, $y_2 = 1$

$x_3 = 2$, $y_3 = 2$



Shape functions using area coordinates:

$N_1 = \zeta_1$, $N_2 = \zeta_2$, $N_3 = \zeta_3$

$$\begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} x_2y_3 - x_3y_2 & y_2 - y_3 & x_3 - x_2 \\ x_3y_1 - x_1y_3 & y_3 - y_1 & x_1 - x_3 \\ x_1y_2 - x_2y_1 & y_1 - y_2 & x_2 - x_1 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix}$$

$x_{jk} = x_j - x_k$

$y_{jk} = y_j - y_k$

↳ describe $\zeta_{1,2,3}$ by cartesian coordinates

The displacement in x,y- direction can be described by

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} \zeta_1 & 0 & \zeta_2 & 0 & \zeta_3 & 0 \\ 0 & \zeta_1 & 0 & \zeta_2 & 0 & \zeta_3 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \mathbb{N}\mathbb{u}^e$$

↳ to find $\mathbb{K}^e$ we need to compute the integral

$$\mathbb{K}^e = \int_{\Omega^e} h \, \mathbb{B}^T \mathbb{E} \, \mathbb{B} \, d\Omega$$

$$= \frac{1}{4A^2} \begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{12} \\ 0 & x_{21} & y_{12} \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix} \int_{\Omega^e} h \, d\Omega$$

$y_{23} = -1, \quad y_{31} = 2, \quad y_{12} = -1 \qquad A = \frac{\sqrt{8} \cdot \sqrt{2}}{2} = \underline{2}$

$x_{32} = -1, \quad x_{13} = -2, \quad x_{21} = 3$

$$\mathbb{E} = \begin{bmatrix} 100 & 25 & 0 \\ 25 & 100 & 0 \\ 0 & 0 & 50 \end{bmatrix}, \quad h = 1$$

$6\times3 \cdot 3\times3 = 6\times3$-matrix

$$= \frac{1}{16} \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 2 & 0 & -2 \\ 0 & -2 & 2 \\ -1 & 0 & 3 \\ 0 & 3 & -1 \end{bmatrix} 25 \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \frac{25}{16} \begin{bmatrix} -4 & -1 & -2 \\ -1 & -4 & -2 \\ 8 & 2 & -4 \\ -2 & -8 & 4 \\ -4 & -1 & 6 \\ 3 & 12 & -2 \end{bmatrix}$$

$h = 1 = \text{constant}$

$$\times \begin{bmatrix} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 3 \\ -1 & -1 & -2 & 2 & 3 & -1 \end{bmatrix} \underbrace{\int_{\Omega_e} h \, d\Omega}_{= \underline{A} = 2}$$

$$= \frac{25}{8} \begin{bmatrix} 6 & 3 & -4 & -2 & -2 & -1 \\ 3 & 6 & 2 & 4 & -5 & -10 \\ -4 & 2 & 24 & -12 & -20 & 10 \\ -2 & 4 & -12 & 24 & 14 & -28 \\ -2 & -5 & -20 & 14 & 22 & -9 \\ -1 & -10 & 10 & -28 & -9 & 38 \end{bmatrix}$$

$K_{11} = \frac{75}{4} = 18{,}75$ ok!

$K_{66} = \frac{475}{4} = 118{,}75$ ok!

2) Sum of row 1,3,5 =

$$[6-4-2 \quad 3+2-5 \quad -4+24-20 \quad -2-12+14 \quad -2-20+22 \quad -1+10-9] = \underline{\underline{[0 \; 0 \; 0 \; 0 \; 0 \; 0]}}$$

Sum of column 1,3,5 =

$$\underline{\underline{\begin{bmatrix} 6-4-2 \\ 3+2-5 \\ -4+24-20 \\ -2-12+14 \\ -2-20+22 \\ -1+10-9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}}$$

If any column or row of a matrix can be written as a linear combination of other columns (rows) then such collection of collumns (rows) is called linearly dependent.

$Row^5/Col^5 = -(Row^1+Row^3) / -(Col^1+Col^3)$

↳ <u>therefore Row^5/Col^5 must vanish</u>

Sum of row 2,4,6:

$$[3-2-1 \quad 6+4-10 \quad 2-12+10 \quad 4+24-28 \quad -5+14-9 \quad -10-28+38] = \Phi \underbrace{[1\times6]}_{\text{size}}$$

Sums of column 2,4,6:

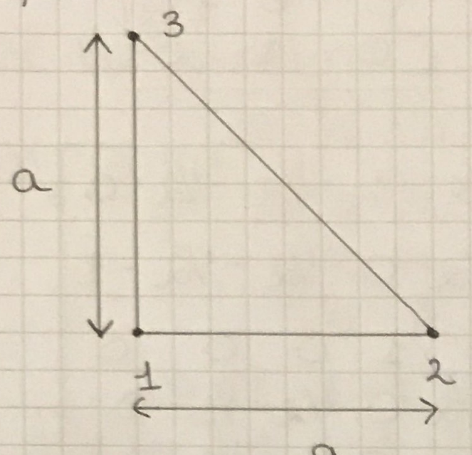
$$\begin{bmatrix} 3-2-1 \\ 6+4-10 \\ 2-12+10 \\ 4+24-28 \\ -5+14-9 \\ -10-28+38 \end{bmatrix} = \Phi \underbrace{[6\times1]}_{\text{size}}$$

⇒ Same reason as for Row/Col 1,35...

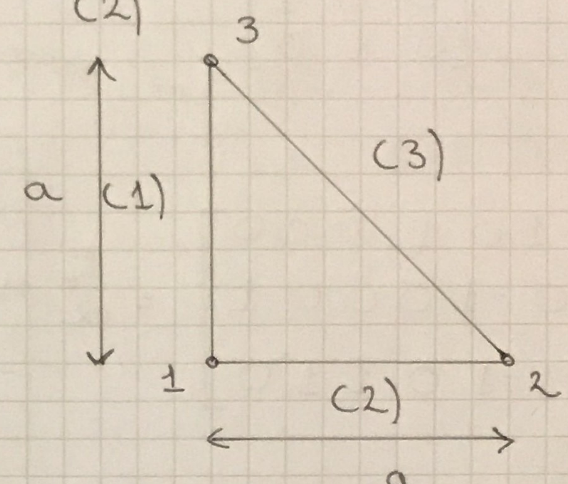
Assignment 3.2

We got two structural models:

(1)



(2)



This a Turner triangle element

A set of three bar elements placed over the edges of the triangular domain.

For both cases: $h = a = 1$.

$\nu = 0$

a) For model (1) we use the same approach as in 3.1 but different (x,y)-coordinates

$A = 1/2$, $x_{21} = 1$, $x_{13} = 0$ $x_{32} = -1$

$y_{12} = 0$, $y_{31} = 1$, $y_{23} = -1$

$$\mathbb{K}^e = \frac{h}{4A} \begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{bmatrix}^T \underbrace{\begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix}}_{E \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

$$\mathbb{K}^e = \frac{1}{2}\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} E \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} = \frac{E}{2}\begin{bmatrix} -1 & 0 & -1/2 \\ 0 & -1 & -1/2 \\ 1 & 0 & 0 \\ 0 & 0 & 1/2 \\ 0 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix} = \frac{E}{2}\begin{bmatrix} 3/2 & 1/2 & -1 & -1/2 & -1/2 & 0 \\ 1/2 & 3/2 & 0 & -1/2 & -1/2 & -1 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ -1/2 & -1/2 & 0 & 1/2 & 1/2 & 0 \\ -1/2 & -1/2 & 0 & 1/2 & 1/2 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \frac{E}{4}\begin{bmatrix} 3 & 1 & -2 & -1 & -1 & 0 \\ 1 & 3 & 0 & -1 & -1 & -2 \\ -2 & 0 & 2 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Model (2):

For bar elements we use:

$$\mathbb{K}^e = \frac{E^e A^e}{L^e}\begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix} \qquad s = \sin\phi, \quad c = \cos\phi$$

For bar element 1: $\phi = \pi/2 \Rightarrow L = 1 \Rightarrow \sin(\pi/2) = 1$, $\cos(\pi/2) = 0$

$$\mathbb{K}^{(1)} = EA^{(1)} \begin{array}{c} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} u_{x1} \\ u_{y1} \\ u_{x3} \\ u_{y3} \end{array}$$

For bar element 2: $\phi = 0,\ L = 1$ $\quad \cos(0) = 1$, $\sin(0) = 0$

$$\mathbb{K}^{(2)} = EA^{(2)} \begin{array}{c} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \end{array}$$

For bar element 3: $\phi = 3\pi/4,\ L = \sqrt{2}$ $\quad \cos\left(\frac{3\pi}{4}\right) = -1/2$, $\sin\left(3\pi/4\right) = 1/2$

$$\mathbb{K}^{(3)} = \frac{\sqrt{2}\cdot EA^{(3)}}{2} \begin{bmatrix} 1/2 & -1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{bmatrix}$$

$$= \frac{\sqrt{2}EA^{(3)}}{4} \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{array}{c} u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{array}$$

Global $\mathbb{K}$:

$$\mathbb{K} = E \cdot \begin{bmatrix} A^{(2)} & 0 & -A^{(2)} & 0 & 0 & 0 \\ 0 & A^{(1)} & 0 & 0 & 0 & -A^{(1)} \\ -A^{(2)} & 0 & A^{(2)} + \frac{\sqrt{2}}{4}A^{(3)} & -\frac{\sqrt{2}}{4}A^{(3)} & -\frac{\sqrt{2}}{4}A^{(3)} & \frac{\sqrt{2}}{4}A^{(3)} \\ 0 & 0 & -\frac{\sqrt{2}}{4}A^{(3)} & \frac{\sqrt{2}}{4}A^{(3)} & \frac{\sqrt{2}}{4}A^{(3)} & -\frac{\sqrt{2}}{4}A^{(3)} \\ 0 & 0 & -\frac{\sqrt{2}}{4}A^{(3)} & \frac{\sqrt{2}}{4}A^{(3)} & \frac{\sqrt{2}}{4}A^{(3)} & -\frac{\sqrt{2}}{4}A^{(3)} \\ 0 & -A^{(1)} & \frac{\sqrt{2}}{4}A^{(3)} & -\frac{\sqrt{2}}{4}A^{(3)} & -\frac{\sqrt{2}}{4}A^{(3)} & \frac{\sqrt{2}}{4}A^{(3)} + A^{(1)} \end{bmatrix}$$

for this case $A_1 = A_2 = A_3$

$$\Rightarrow \frac{E}{4} \begin{bmatrix} 4 & 0 & -4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & -4 \\ -4 & 0 & 4+\sqrt{2} & -\sqrt{2} & -\sqrt{2} & \sqrt{2} \\ 0 & 0 & -\sqrt{2} & \sqrt{2} & \sqrt{2} & -\sqrt{2} \\ 0 & 0 & -\sqrt{2} & \sqrt{2} & \sqrt{2} & -\sqrt{2} \\ 0 & -4 & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & 4+\sqrt{2} \end{bmatrix}$$

we can see the $\mathbb{K}_{bar}$ is not equivalent to the $\mathbb{K}_{triang}$.

b) $A_1 = A_2 = A \Rightarrow A_3 = A'$

$$\mathbb{K}_{bar} = \frac{E}{4} \begin{bmatrix} 4A & 0 & -4A & 0 & 0 & 0 \\ 0 & 4A & 0 & 0 & 0 & -4A \\ -4A & 0 & 4A+\sqrt{2}A' & -\sqrt{2}A' & -\sqrt{2}A' & \sqrt{2}A' \\ 0 & 0 & -\sqrt{2}A' & \sqrt{2}A' & \sqrt{2}A' & -\sqrt{2}A' \\ 0 & 0 & -\sqrt{2}A' & \sqrt{2}A' & \sqrt{2}A' & -\sqrt{2}A' \\ 0 & -4A & \sqrt{2}A' & -\sqrt{2}A' & -\sqrt{2}A' & 4A+\sqrt{2}A' \end{bmatrix}$$

$$\mathbb{K}_{triang} = \frac{E}{4} \begin{bmatrix} 3 & 1 & -2 & -1 & -1 & 0 \\ 1 & 3 & 0 & -1 & -1 & -2 \\ -2 & 0 & 2 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \end{bmatrix}$$

As we can see it is impossible to make them equivalent. If we put $A' = \frac{1}{\sqrt{2}}$ and $A = 1/4$

we get the 4 last diagonal terms to be equal. If we put $A = 3/4$ then K_{11} and K_{22} are correct, but then K_{33} and K_{66} is wrong.

c) The bar/truss element are three bars connected to shape triangle (with no interior). The Turner's triangle is a continuous triangle element with 3 nodes.

These are two different systems and will not not be able to provide equivalent stiffness matrix.

d) If $\nu \neq 0$

$$\Rightarrow \mathbb{E} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

$\Rightarrow \mathbb{K}_{bar}$ will be the same because it is not dependent on the Poisson ratio ν.

$$\mathbb{K}_{\text{triangle}} = \frac{1}{2}\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \times \begin{bmatrix} \cdots \end{bmatrix}$$

$$= \frac{E}{2(1-\nu^2)}\begin{bmatrix} -1 & -\nu & \frac{\nu-1}{2} \\ -\nu & -1 & \frac{\nu-1}{2} \\ 1 & \nu & 0 \\ 0 & 0 & \frac{1-\nu}{2} \\ 0 & 0 & \frac{1-\nu}{2} \\ \nu & 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$= \frac{E}{2(1-\nu^2)}\begin{bmatrix} 1+\frac{1-\nu}{2} & \frac{1+\nu}{2} & -1 & \frac{\nu-1}{2} & \frac{\nu-1}{2} & -\nu \\ \frac{1+\nu}{2} & 1+\frac{1-\nu}{2} & -\nu & \frac{\nu-1}{2} & \frac{\nu-1}{2} & -1 \\ -1 & -\nu & 1 & 0 & 0 & \nu \\ \frac{\nu-1}{2} & \frac{\nu-1}{2} & 0 & \frac{1-\nu}{2} & \frac{1-\nu}{2} & 0 \\ \frac{\nu-1}{2} & \frac{\nu-1}{2} & 0 & \frac{1-\nu}{2} & \frac{1-\nu}{2} & 0 \\ -\nu & -1 & \nu & 0 & 0 & 1 \end{bmatrix}$$