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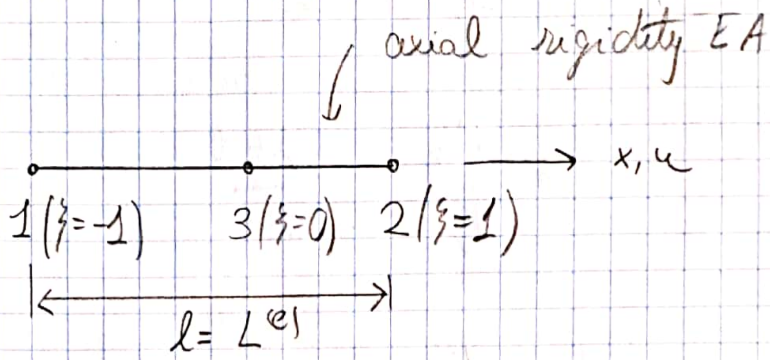
Assignment 4.1



where,

$$x_1 = 0 \quad , \quad x_2 = l \quad , \quad x_3 = \left(\frac{1}{2} + \alpha\right) l$$

$$-\frac{1}{2} < \alpha < \frac{1}{2} \quad , \text{ if } \alpha = 0 \Rightarrow \text{node 3 located at } x = \frac{1}{2} l$$

Isoparametric definition of the element:

$$\begin{bmatrix} 1 \\ x \\ u \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ N_3^e \end{bmatrix}$$

1. Get the Jacobian $J = \frac{dx}{d\xi}$ in terms of l, α and ξ.

* Position of the nodes:

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3$$

* The Jacobian:

$$J = \frac{dx}{d\xi} = \frac{dN_1}{d\xi} x_1 + \frac{dN_2}{d\xi} x_2 + \frac{dN_3}{d\xi} x_3$$

\+ The shape functions are:

$$N_1 = \frac{\xi(-1+\xi)}{2} \rightarrow \frac{dN_1}{d\xi} = \frac{2\xi-1}{2}$$

$$N_2 = \frac{\xi(1+\xi)}{2} \rightarrow \frac{dN_2}{d\xi} = \frac{2\xi+1}{2}$$

$$N_3 = (1-\xi^2) \rightarrow \frac{dN_3}{d\xi} = -2\xi$$

\+ Then, the Jacobian:

$$J = \frac{dx}{d\xi} = \frac{2\xi-1}{2} \cdot 0 + \frac{2\xi+1}{2} L + -2\xi\left(\frac{L}{2} + \alpha L\right) =$$

$$\left[J = L\left(\frac{1}{2} - 2\alpha\xi\right)\right]$$

\- Show, if $-1/4 < \alpha < 1/4$, then $J>0$ over the whole element.

The Jacobian will be positive when: $\frac{1}{2} - 2\alpha\xi > 0$,

as L (length) is always positive

\# So, the parameters that influence the result are α and ξ.

\# If ξ is negative : $-1 < \xi < 0$;

$\Rightarrow \frac{1}{2} + 2\alpha\xi$

- If $\alpha > 0 \Rightarrow$ Jacobian would be positive when $\alpha < \frac{1}{4}$

$\Rightarrow \frac{1}{2} + 2\alpha\xi > 0$; $\alpha > -\frac{1}{4\xi} \xrightarrow{\xi=-1} \left[\alpha < \frac{1}{4}\right]$

- If $\alpha \leq 0$

$\Rightarrow \frac{1}{2} - 2\alpha\xi > 0$; $\alpha < \frac{1}{4\xi} \xrightarrow{\xi=-1} \left[\alpha > -\frac{1}{4}\right]$

If $\alpha > -\frac{1}{4}$, then $J > 0$

\# If ξ is positive : $0 < \xi < 1$

$\Rightarrow \frac{1}{2} - 2\alpha\xi$

- If $\alpha > 0$

$\frac{1}{2} - 2\alpha\xi > 0$; $\alpha < \frac{1}{4\xi} \xrightarrow{\xi=1} \left[\alpha < \frac{1}{4}\right]$

If $\alpha < \frac{1}{4}$, then $J > 0$

· If $\alpha \leq 0$

$$\frac{1}{2} + 2\alpha\xi > 0 \rightarrow \left[\alpha > -\frac{1}{4}\right]$$

In conclusion,

$$J > 0 \text{ when } \forall \alpha \in \left[-\frac{1}{4}, \frac{1}{4}\right]$$

- Show if $\alpha = 0$, $J = \frac{1}{2}$ is a constant over the element.

Substituting into the value of the Jacobian:

$$J = \frac{dx}{d\xi} = L\left(\frac{1}{2} - 2\alpha\xi\right) \xrightarrow{\alpha = 0} \left[J = \frac{L}{2}\right]$$

Therefore, it does not depend on any parameter, it is a constant over the element

2. Obtain B, function of 1, α and ξ

$$B = \frac{dN}{dx} = J^{-1}\frac{dN}{d\xi} = \frac{1}{J}\begin{bmatrix} \frac{dN_1}{d\xi} \\ \frac{dN_2}{d\xi} \\ \frac{dN_3}{d\xi} \end{bmatrix} = \frac{1}{L\left(\frac{1}{2} - 2\alpha\xi\right)}\begin{bmatrix} \frac{2\xi - 1}{2} \\ \frac{2\xi + 1}{2} \\ -2\xi \end{bmatrix}$$

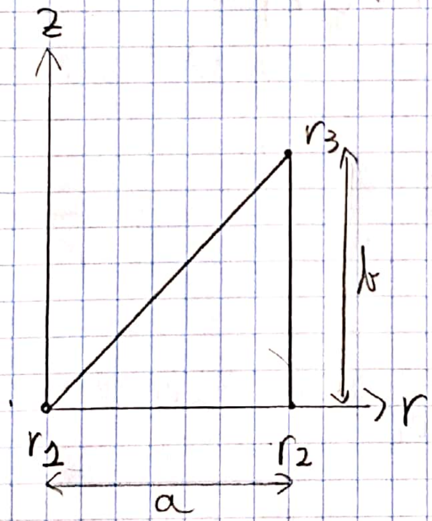
Assignment 4.2

1 Compute K^e for axisymmetric triangle

$r_1 = 0$, $r_2 = r_3 = a$, $z_1 = z_2 = 0$, $z_3 = b$

$\nu = 0$ and stress-strain matrix:

$$\underline{E} = E \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$$



\# The element stiffness matrix of an axisymmetric linear triangular element is:

$$\underline{K}^e = 2\pi \int_A \underline{r}\ \underline{B}^T \underline{E}\ \underline{B}\ dA$$

Where $\underline{B}$ is:

$$\underline{B} = \frac{1}{2A^{(e)}} \begin{bmatrix} \frac{\partial N_i}{\partial r} & 0 \\ 0 & \frac{\partial N_i}{\partial z} \\ \frac{N_i}{r} & 0 \\ \frac{\partial N_i}{\partial z} & \frac{\partial N_i}{\partial r} \end{bmatrix}$$

* The shape functions of the axisymmetric triangle are

$$N_i = \frac{1}{2A^{(e)}} (a_i + b_i r + c_i z)$$

where:

$$a_i = r_j z_k - r_k z_j$$
$$b_i = z_j - z_k$$
$$c_i = r_k - r_j$$

In order to calculate B_i:

Nodes	r	z	a_i	b_i	c_i
1	0	0	ab	-b	0
2	a	0	0	b	-a
3	a	b	0	0	a

#

$$N_1 = \frac{1}{2A}(ab - br) = 1 - \frac{r}{a}$$

$$N_2 = \frac{1}{2A}(br - az) = \frac{r}{a} - \frac{z}{b}$$

$$N_3 = \frac{1}{2A}(az) = \frac{z}{b}$$

The area of the triangle is $A = \frac{ab}{2}$

So, the matrix B is:

$$\underline{B_i} = \left[\begin{array}{cc|cc|cc} -1/a & 0 & \frac{1}{a} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{b} & 0 & \frac{1}{b} \\ \frac{a-r}{ar} & 0 & \frac{br-az}{abr} & 0 & \frac{z}{br} & 0 \\ 0 & -\frac{1}{a} & -\frac{1}{b} & \frac{1}{a} & \frac{1}{b} & 0 \end{array}\right]$$

* In order to simplify the calculations, it is considered $\underline{r}$ and $\underline{z}$ as

$$\underline{r_c} = \frac{r_1 + r_2 + r_3}{3} = \frac{2}{3}a$$

$$\underline{z_c} = \frac{z_1 + z_2 + z_3}{3} = \frac{1}{3}b$$

Once everything is computed; with the help of Matlab:

$$\underline{K}^e = 2\pi\, \underline{B}^T\, \underline{E}\, \underline{B}\, r_c \cdot A$$

$$\underline{K}^{(e)} = \frac{2\pi E}{3b} \begin{bmatrix} 5b^2/4 & 0 & -3b^2/4 & 0 & b^2/4 & 0 \\ & b^2/2 & ab/2 & -b^2/2 & -ab/2 & 0 \\ & & a^2/2 + 5b^2/4 & -ab/2 & b^2/4 - a^2/2 & 0 \\ & & & a^2 + b^2/2 & (ab)/2 & -a^2 \\ & \text{Sym} & & & a^2/2 + b^2/4 & 0 \\ & & & & & a^2 \end{bmatrix}$$

2. Show sum of the rows (and colums) 2, 4 and 6 of $K^{(e)}$, must vanish and why

" " 1, 3 and 5 not vanish and why

+ As it is easily checked, the sum of rows and colums 2, 4 and 6 of K^e are equal to 0. This is because the stiffness matrix is unconstrained, the nodes are free to move and there is rigid body motion due to the rigidity of those degrees of freedom (equal to 0) in the z direction components

+ As it is easily checked, the sum of rows and colums 1, 3 and 5 are not equal to 0.

These rows and columns correspond to the degrees of freedom in r direction and the result of the sum is different to 0.

This fact is produced because of the intrinsic restriction in the matrix due to the axisymmetric condition.
This means that the nodes are not allowed to move completely free through the r axis because there is an stiffness imposed.

3 Compute the consistent force vector $f^{(e)}$ for gravity forces $\underline{b} = [0, -g]^T$.

+ To calculate the forces:

$$\{f_b\} = 2\pi \int_A r \, \underline{N}^T \cdot \underline{b} \, dA.$$

+ For each node

$$f_{bi} = \begin{bmatrix} f_{bir} \\ f_{biz} \end{bmatrix} = \frac{2\pi A r_c}{3} \begin{pmatrix} 0 \\ b_{zi} \end{pmatrix}$$

+ Finally,

$$\left[f^{(e)} = \frac{2\pi a^2 b}{9} \begin{pmatrix} 0 \\ -g \\ 0 \\ -g \\ 0 \\ -g \end{pmatrix} \right]$$