CSMD

Assignment-04

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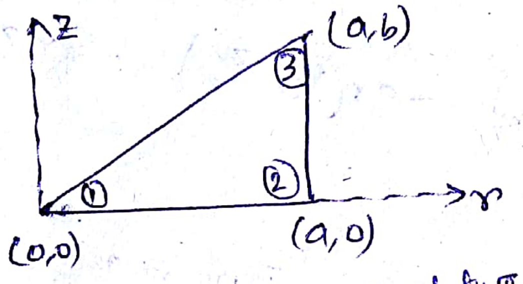
4.1/ Given, for an axisymmetric triangle,

$r_1 = 0, \; r_2 = r_3 = a, \; z_1 = z_2 = 0, \; z_3 = b$



Material is isotropic with $\nu = 0$ &

$$E = E \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$$

We define linear shape functions, N_1, N_2 & N_3 as follows :-

$N_1 = \frac{1}{2A}(f_1 + r b_1 + z c_1)$; $N_2 = \frac{1}{2A}(f_2 + r b_2 + z c_2)$

& $N_3 = \frac{1}{2A}(f_2 + r b_2 + z c_2)$

And,

$f_1 = r_2 z_3 - r_3 z_2 = ab - 0 = ab$	$b_1 = z_2 - z_3 = -b$	$c_1 = r_3 - r_2 = 0$
$f_2 = r_3 z_1 - r_1 z_3 = 0 - 0 = 0$	$b_2 = z_3 - z_1 = b$	$c_2 = r_1 - r_3 = -a$
$f_3 = r_1 z_2 - r_2 z_1 = 0 - 0 = 0$	$b_3 = z_1 - z_2 = 0$	$c_3 = r_2 - r_1 = a$

& $A = \frac{1}{2} ab$ [from geometry of triangle]

$\therefore \; N_1 = 1 - r/a$.

$N_2 = r/a - z/b$

$N_3 = z/b$

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②

Now,

The stiffness matrix can be evaluated using a simple one point integration,

$$K = 2\pi \bar{r} A \bar{B} E \bar{B}^T \quad \text{— ②}$$

where, $\bar{B} = B(\bar{r}, \bar{z})$, $(\bar{r}, \bar{z})$ is the coordinate of the centroid of our elemental triangle.

$$\bar{r} = \frac{r_1 + r_2 + r_3}{3} = \frac{2}{3} a$$

$$\bar{z} = \frac{z_1 + z_2 + z_3}{3} = \frac{1}{3} b$$

∴ From ①, we have (at the centroid),

$$N_1 = 1 - \frac{\bar{r}}{a} = \frac{1 - (2/3)a}{a} = \frac{1}{3}$$

$$N_2 = \frac{\bar{r}}{a} - \frac{\bar{z}}{b} = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

$$N_3 = \frac{\bar{z}}{b} = \frac{1}{3}$$

Area of the triangle, $A = \frac{1}{2} \begin{vmatrix} r_1 & r_2 & r_3 \\ z_1 & z_2 & z_3 \\ 1 & 1 & 1 \end{vmatrix} = \frac{1}{2} \begin{vmatrix} 0 & a & a \\ 0 & 0 & b \\ 1 & 1 & 1 \end{vmatrix}$

$= \frac{1}{2} ab$, which can also be seen from the geometry.

$$\therefore \frac{2AN_1}{\bar{r}} = \frac{2AN_2}{\bar{r}} = \frac{2AN_3}{\bar{r}} = \frac{2 \times \frac{1}{2} ab \times \frac{1}{3}}{\frac{2}{3} a} = \frac{ab/3}{2a/3} = b/2$$

Now,

$$\bar{B}^T = \frac{1}{2A}\begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ \frac{2AN_1}{r} & 0 & \frac{2AN_2}{r} & 0 & \frac{2AN_3}{r} & 0 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix}$$

Substituting the values, we have,

$$\bar{B}^T = \frac{1}{2\times(\frac{1}{2}ab)}\begin{bmatrix} -b & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & -a & 0 & a \\ b/2 & 0 & b/2 & 0 & b/2 & 0 \\ 0 & -b & -a & b & a & 0 \end{bmatrix}$$

$\therefore$ From (2), we get,

$$K = 2\pi \bar{r} A \bar{B} E \bar{B}^T$$

$$= 2\pi \tfrac{1}{2}(ab)\times\tfrac{2}{3}a\, \bar{B} E \bar{B}^T$$

$$= \frac{2}{3}a^2 b\pi\, \bar{B} E \bar{B}^T \quad \text{--- (3)}$$

Now,

$\bar{B} E \bar{B}^T$

$$= \frac{E}{4A^2}\begin{bmatrix} -b & 0 & b/2 & 0 \\ 0 & 0 & 0 & -b \\ b & 0 & b/2 & -a \\ 0 & -a & 0 & b \\ 0 & 0 & b/2 & a \\ 0 & a & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} -b & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & -a & 0 & a \\ b/2 & 0 & b/2 & 0 & b/2 & 0 \\ 0 & -b & -a & b & a & 0 \end{bmatrix}$$

(4)

$$= \frac{E}{4A^2}\begin{bmatrix} -b & 0 & b/2 & 0 \\ 0 & 0 & 0 & -b \\ b & 0 & b/2 & -a \\ 0 & -a & 0 & b \\ 0 & 0 & b/2 & a \\ 0 & a & 0 & 0 \end{bmatrix}\begin{bmatrix} -b & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & -a & 0 & a \\ b/2 & 0 & b/2 & 0 & b/2 & 0 \\ 0 & -b/2 & -a/2 & b/2 & a/2 & 0 \end{bmatrix}$$

$$= \frac{E}{4(\frac{1}{2}a^2b^2)}\begin{bmatrix} \frac{5}{4}b^2 & 0 & -\frac{3}{4}b^2 & 0 & \frac{1}{4}b^2 & 0 \\ 0 & \frac{1}{2}b^2 & \frac{1}{2}ab & -\frac{1}{2}b^2 & -\frac{1}{2}ab & 0 \\ -\frac{3}{4}b^2 & \frac{1}{2}ab & (\frac{5}{4}b^2+\frac{1}{2}a^2) & -\frac{1}{2}ab & (\frac{1}{4}b^2-\frac{1}{2}a^2) & 0 \\ 0 & -\frac{1}{2}b^2 & -\frac{1}{2}ab & (\frac{1}{2}b^2+a^2) & \frac{1}{2}ab & -a^2 \\ \frac{1}{4}b^2 & \frac{1}{2}ab & (\frac{1}{4}b^2-\frac{1}{2}a^2) & \frac{1}{2}ab & (\frac{1}{4}b^2+\frac{1}{2}a^2) & 0 \\ 0 & 0 & 0 & -a^2 & 0 & a^2 \end{bmatrix}_{\text{symm}}$$

Now, from (3) we have,

$$K = \boxed{\frac{2}{3}\cancel{a^2}\cancel{b}\pi\frac{E}{\cancel{a^2}b^{\cancel{2}}}}\begin{bmatrix} \frac{5}{4}b^2 & 0 & -\frac{3}{4}b^2 & 0 & \frac{1}{4}b^2 & 0 \\ & \frac{1}{2}b^2 & \frac{1}{2}ab & -\frac{1}{2}b^2 & -\frac{1}{2}ab & 0 \\ & & (\frac{5}{4}b^2+\frac{1}{2}a^2) & -\frac{1}{2}ab & (\frac{1}{4}b^2-\frac{1}{2}a^2) & 0 \\ & & & (\frac{1}{2}b^2+a^2) & \frac{1}{2}ab & -a^2 \\ & & & & (\frac{1}{4}b^2+\frac{1}{2}a^2) & 0 \\ & & & & & a^2 \end{bmatrix}_{\text{symm}}$$

$$= \frac{2\pi E}{3b}$$

(5) ∴ The stiffness matrix equation equation takes the form -

$$KU = f$$

$$\Rightarrow \frac{2\pi E}{3b}\begin{bmatrix} \frac{5}{4}b^2 & 0 & -\frac{3}{4}b^2 & 0 & \frac{1}{4}b^2 & 0 \\ \cdots & \frac{1}{2}b^2 & \frac{1}{2}ab & -\frac{1}{2}b^2 & -\frac{1}{2}ab & 0 \\ \cdots & \cdots & \left(\frac{5}{4}b^2+\frac{1}{2}a^2\right) & -\frac{1}{2}ab & \left(\frac{1}{4}b^2-\frac{1}{2}a^2\right) & 0 \\ \cdots & \cdots & \cdots & \left(\frac{1}{2}b^2+a^2\right) & \frac{1}{2}ab & -a^2 \\ \cdots & \cdots & \cdots & \cdots & \left(\frac{1}{4}b^2+\frac{1}{2}a^2\right) & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & a^2 \end{bmatrix} \begin{bmatrix} U_{r_1} \\ U_{z_1} \\ U_{r_2} \\ U_{z_2} \\ U_{r_3} \\ U_{z_3} \end{bmatrix} = \begin{bmatrix} f_{r_1} \\ f_{z_1} \\ f_{r_2} \\ f_{z_2} \\ f_{r_3} \\ f_{z_3} \end{bmatrix}$$

(2)

Since the object is axisymmetric, for revolution the deformation will only be limited to radial direction (r). in the above equation.

Hence, $U_{z_1} = U_{z_2} = U_{z_3} = 0$ in the above equation.

So, the rows & columns 2, 4 & 6 of 'K' must vanish.

But U_{r_1}, U_{r_2} & U_{r_3} are not equal to zero, the other rows & columns must exist in the equation.

3/

The force vector can be expressed as follows -

$$f = \int_A N^T b \, r \, dA \quad \text{--- (4)}$$

Now,
$$r = \sum_{i=1}^{3} r_i N_i$$

$$= r_1 N_1 + r_2 N_2 + r_3 N_3$$

$$= r - \frac{2a}{b} + \frac{2a}{b}$$

$$= r$$

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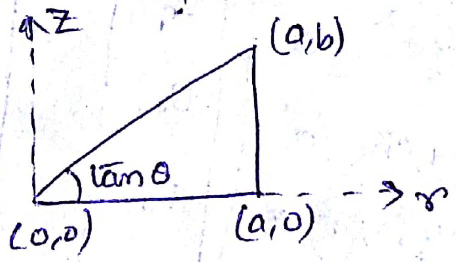
Now, from 4,

$$f = \int_r \int_Z F \, dz \, dr \quad , \quad F = N^T b r \qquad \text{——— } (5)$$

Here, r varies from 0 to a as can be seen from the geometry of the triangular element.



From the geometry, we find the following relation between r & Z -

$$Z/r = \tan\theta = b/a$$

$$\Rightarrow Z = br/a$$

$\therefore$ Z varies from 0 to br/a.

Now,

$$F = N^T b r$$

$$= r \begin{bmatrix} (1 - r/a) & 0 \\ (r/a - z/b) & 0 \\ z/b & 0 \\ 0 & 1 - r/a \\ 0 & r/a - z/b \\ 0 & z/b \end{bmatrix}_{6\times 2} \begin{bmatrix} 0 \\ -g \end{bmatrix}_{2\times 1}$$

⑦

$$= \begin{bmatrix} 0 & 0 & 0 & -gr(1-r/a) & -gr(r/a - z/b) & -grz/b \end{bmatrix}$$

∴ From ⑤,

$$f = \int_0^a \int_0^{br/a} \begin{bmatrix} 0 & 0 & 0 & -gr(1-r/a) & -gr(r/a - z/b) & -grz/b \end{bmatrix} dz\,dr \quad \text{—⑥}$$

Integrating the above equation term by term we get.

$$\int_0^a \int_0^{br/a} -gr(1-r/a)\,dz\,dr = -g\frac{a^2 b}{12}$$

$$\int_0^a \int_0^{br/a} -gr(r/a - z/b)\,dz\,dr = \frac{-gba^2}{8}, \text{ \&}$$

$$\int_0^a \int_0^{br/a} -gr\,z/b\,dz\,dr = -\frac{gba^2}{8}$$

$$\int_0^a \int_0^{br/a} 0\,dz\,dr = 0 \quad [\because \text{definite integral of } 0 \text{ gives } 0]$$

∴ From ⑥, we have

$$f = -g\frac{a^2 b}{12}\begin{bmatrix} 0 & 0 & 0 & 1 & 3/2 & 3/2 \end{bmatrix}.\ \text{per unit density.}$$