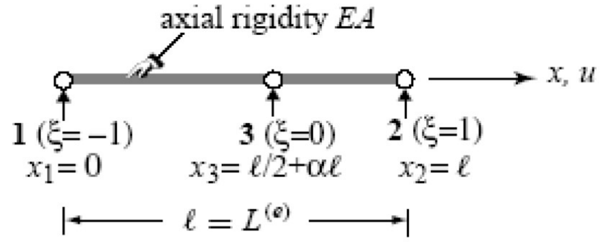
Assignment 4.1



1)

$$N_1(\xi) = \frac{1}{2}\xi(\xi-1)$$
$$N_3(\xi) = 1-\xi^2$$
$$N_2(\xi) = \frac{1}{2}\xi(1+\xi)$$

$$x = \sum_{i=1}^{3} x_i N_i = \frac{1}{2}\xi(\xi-1)\cdot 0 + \frac{L}{2}\xi(1+\xi) + \left(\frac{1}{2}+\alpha\right)L\cdot(1-\xi^2) =$$
$$= -\alpha L\xi^2 + \frac{L}{2}\xi + L\left(\frac{1}{2}+\alpha\right)$$

$$J = \frac{dx}{d\xi} = -2\alpha L\xi + \frac{L}{2} = L\left(\frac{1}{2} - 2\alpha\xi\right)$$

if $-1/4 < \alpha < 1/4$ | $\alpha = -1/4 \Rightarrow J = L\left(\frac{1}{2}+\frac{\xi}{2}\right)$; $\alpha = 1/4 \Rightarrow J = L\left(\frac{1}{2}-\frac{\xi}{2}\right)$; $-1 \le \xi \le 1 \downarrow J > 0$

if $\alpha = 0 \rightarrow J = \frac{L}{2}$ ok!

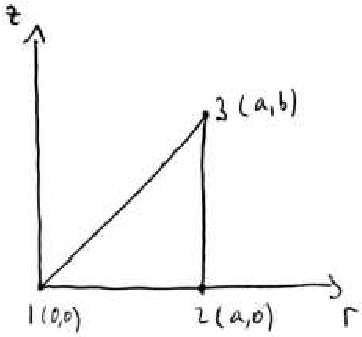
2)

$$B = \frac{dN}{dx} = \frac{1}{J}\cdot\frac{dN}{d\xi} = \frac{1}{L\left(\frac{1}{2}-2\alpha\xi\right)}\left[\frac{dN_1}{d\xi},\ \frac{dN_2}{d\xi},\ \frac{dN_3}{d\xi}\right] =$$
$$= \frac{1}{L\left(\frac{1}{2}-2\alpha\xi\right)}\left[\xi-\frac{1}{2},\ \xi+\frac{1}{2},\ -2\xi\right]$$

$$e = \frac{du}{dx} = Bu^e = \frac{1}{L\left(\frac{1}{2}-2\alpha\xi\right)}\left[\xi-\frac{1}{2},\ \xi+\frac{1}{2},\ -2\xi\right]\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}$$

Assignment 4.2

1)



$$A = \frac{ab}{2} \qquad \nu = 0 \rightarrow E = E\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ r \\ z \\ u_r \\ u_z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & a & a \\ 0 & 0 & b \\ u_{r_1} & u_{r_2} & u_{r_3} \\ u_{z_1} & u_{z_2} & u_{z_3} \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix}$$

$$\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} r_2 z_3 - z_2 r_3 & z_{23} & r_{32} \\ r_3 z_1 - z_3 r_1 & z_{31} & r_{13} \\ r_1 z_2 - z_1 r_2 & z_{12} & r_{21} \end{bmatrix} \begin{bmatrix} 1 \\ r \\ z \end{bmatrix}$$

$$\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \frac{1}{ab} \begin{bmatrix} ab & -b & 0 \\ 0 & b & -a \\ 0 & 0 & a \end{bmatrix} \begin{bmatrix} 1 \\ r \\ z \end{bmatrix}$$

$$N_1 = \frac{1}{ab}(ab - br)$$

$$N_2 = \frac{1}{ab}(br - az)$$

$$N_3 = \frac{1}{ab}\, az$$

$$N = \begin{pmatrix} N_1 & N_2 & N_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & N_1 & N_2 & N_3 \end{pmatrix}$$

$$D = \begin{pmatrix} \partial/\partial r & 0 \\ 0 & \partial/\partial z \\ 1/r & 0 \\ \partial/\partial z & \partial/\partial r \end{pmatrix}$$

$$B = DN = \begin{pmatrix} q_r & 0 \\ 0 & q_z \\ q_\theta & 0 \\ q_z & q_r \end{pmatrix}$$

$$q_r = \left[\frac{\partial N_1}{\partial r}, \frac{\partial N_2}{\partial r}, \frac{\partial N_3}{\partial r} \right] = \left[-\frac{1}{a}, \frac{1}{a}, 0 \right]$$

$$q_z = \left[\frac{\partial N_1}{\partial z}, \frac{\partial N_2}{\partial z}, \frac{\partial N_3}{\partial z} \right] = \left[0, -\frac{1}{b}, \frac{1}{b} \right]$$

$$q_\theta = \left[\frac{N_1}{r}, \frac{N_2}{r}, \frac{N_3}{r} \right] = \left[\frac{1}{r} - \frac{1}{a}, \frac{1}{a} - \frac{z}{br}, \frac{z}{br} \right]$$

$$B = \begin{bmatrix} -1/a & 1/a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1/b & 1/b \\ \frac{1}{r} - \frac{1}{a} & \frac{1}{a} - \frac{z}{br} & \frac{z}{br} & 0 & 0 & 0 \\ 0 & -1/b & 1/b & -1/a & 1/a & 0 \end{bmatrix}$$

$$\mathbb{K}^e = \int_V B^T \cdot E \cdot B \, dV = \int_V B^T E \cdot B \, r \, dr \cdot d\theta \cdot dz =$$

$$= 2\pi \iint B^T \cdot E \cdot B \cdot r \cdot dr \cdot dz = \quad \begin{array}{l} 0 \leq r \leq a \\ 0 \leq z \leq \frac{b}{a} r \end{array}$$

$$= 2\pi \int_0^a \int_0^{br/a} B^T \cdot E \cdot B \, r \, dz \, dr$$

$$\mathbb{K}^e = 2\pi E \begin{bmatrix} 2b/3 & -b/4 & b/12 & 0 & 0 & 0 \\ -b/4 & \frac{a^2}{6b}+\frac{4b}{9} & \frac{-a^2}{6b}+\frac{b}{18} & \frac{a}{6} & \frac{-a}{6} & 0 \\ b/12 & \frac{-a^2}{6b}+\frac{b}{18} & \frac{a^2}{6b}+\frac{b}{9} & \frac{-a}{6} & \frac{a}{6} & 0 \\ 0 & \frac{a}{6} & \frac{-a}{6} & \frac{b}{6} & \frac{-b}{6} & 0 \\ 0 & -\frac{a}{6} & \frac{a}{6} & -\frac{b}{6} & \frac{a^2}{3b}+\frac{b}{6} & \frac{-a^2}{3b} \\ 0 & 0 & 0 & 0 & \frac{-a^2}{3b} & \frac{a^2}{3b} \end{bmatrix}$$

2)

In this case the columns 4,5 and 6 corresponds the translation of the element in the z direction. In this direction, the model is not able to produce a rigid body displacement that will not generate stresses. Only the relative movements between these nodes generate stresses, when all of them are equal, the resultant force is zero.
On the other hand, 1, 2 and 3 columns respond to translation of the element in r direction. The prevented movement due to the antisymmetry of the model, means that if there is movement in this direction it will generate forces.

$$\begin{pmatrix} 2b/3 \\ -b/4 \\ b/12 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -b/4 \\ \frac{a^2}{6b}+\frac{4b}{9} \\ \frac{-a^2}{6b}+\frac{b}{18} \\ a/6 \\ -a/6 \\ 0 \end{pmatrix} + \begin{pmatrix} b/12 \\ \frac{-a^2}{6b}+\frac{b}{18} \\ \frac{a^2}{6b}+\frac{b}{9} \\ -a/6 \\ a/6 \\ 0 \end{pmatrix} = \begin{pmatrix} b/2 \\ b/4 \\ b/4 \\ 0 \\ 0 \\ 0 \end{pmatrix} \neq 0$$

Column 1 Column 2 Column 3

$$\begin{pmatrix} 0 \\ a/6 \\ -a/6 \\ b/6 \\ -b/6 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -a/6 \\ a/6 \\ -b/6 \\ \frac{a^2}{3b}+\frac{b}{6} \\ -\frac{a^2}{3b} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\frac{a^2}{3b} \\ \frac{a^2}{3b} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0$$

Column 4

3)

$$f^e = \int_V N^T \cdot b \, dV = 2\pi \int_A N^T b \cdot r \cdot dr d\theta$$

$$f^e = 2\pi \int_0^a \int_0^{br/a} \begin{bmatrix} N_1 & 0 \\ N_2 & 0 \\ N_3 & 0 \\ 0 & N_1 \\ 0 & N_2 \\ 0 & N_3 \end{bmatrix} \begin{bmatrix} 0 \\ -g \end{bmatrix} r \, dr \, dz =$$

$$= 2\pi \int_0^a \int_0^{br/a} \frac{g}{ab} \cdot \begin{bmatrix} ab-br & 0 \\ br-az & 0 \\ az & 0 \\ 0 & ab-br \\ 0 & br-az \\ 0 & az \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \end{bmatrix} r \, dr \, dz$$

$$f^e = 2\pi g a^2 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1/12 \\ (b+1/2)/45 \\ -1/8b \end{pmatrix}$$