CSMD Assignemnt 4

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1 Question 1

The stiffness matrix of an axisymmetric triangle is given by

$$\mathbf{K} = 2\pi \int_A r\mathbf{B^T E B}\, dr\, dz$$

where

$$\mathbf{B} = \mathbf{DN} = \begin{bmatrix} \frac{\partial}{\partial r} & 0 \\ 0 & \frac{\partial}{\partial z} \\ \frac{1}{r} & 0 \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial r} \end{bmatrix} \begin{bmatrix} N1 & 0 & N2 & 0 & N3 & 0 \\ 0 & N1 & 0 & N2 & 0 & N3 \end{bmatrix}$$

The constitutive matrix with $\nu = 0$ is given as

$$\mathbf{E} = E \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix}$$

$$N_i = \frac{1}{2A}(\alpha_i + \beta_i r + \gamma_i z)$$

where i goes from 1 to 3, A is the area, and the constants α, β, γ are defined from the nodal locations given in the problem.

To solve the integral, we can either transform to natural coordinates and then use Gauss quadratures, or to simplify the problem we can substitute r and z by the mid point. This was solved using MATLAB; code can be found in the Appendix.

$$K = \frac{2E\pi}{3b} \begin{bmatrix} \frac{5b^2}{4} & 0 & \frac{-3b^2}{4} & 0 & \frac{b^2}{4} & 0 \\ 0 & \frac{b^2}{2} & \frac{ab}{2} & \frac{-b^2}{2} & \frac{-ab}{2} & 0 \\ \frac{-3b^2}{4} & \frac{ab}{2} & \frac{5b^2}{4} + \frac{a^2}{2} & \frac{-ab}{2} & \frac{b^2}{4} - \frac{a^2}{2} & 0 \\ 0 & \frac{-b^2}{2} & \frac{-ab}{2} & \frac{b^2}{2} + \frac{a^2}{1} & \frac{ab}{2} & -a^2 \\ \frac{b^2}{4} & \frac{-ab}{2} & \frac{b^2}{4} - \frac{a^2}{2} & \frac{ab}{2} & \frac{b^2}{4} + \frac{a^2}{2} & 0 \\ 0 & 0 & 0 & -a^2 & 0 & a^2 \end{bmatrix}$$

As can be noticed, the sum of the rows and columns 2, 4, 6 vanish. That's because the displacement in the z-direction is just rigid body motion without any energy dissipation.

However, the sum of the rows and column 1, 3, 5 doesn't vanish. That's because a deformation in the r-direction means that the revolved solid is deforming leading to energy dissipation.

The force vector due to body force $b = [0, -g]^T$ is given by

$$\mathbf{F} = 2\pi \int_A \mathbf{N^T b} r\, dr\, dz$$

evaluated at the centroid can be calculated as

$$\mathbf{F} = \frac{2\pi a^2 b}{9} \begin{bmatrix} 0 \\ -g \\ 0 \\ -g \\ 0 \\ -g \end{bmatrix}$$

2 Question 2

Using the line product method we obtain N_5 first.

$$\begin{aligned} N_5 &= C_5 L_{12} L_{23} L_{34} L_{41} \\ &= C_5(\zeta + 1)(\eta - 1)(\zeta - 1)(\eta + 1) \end{aligned}$$

to get C_5 the equation is applied at (0,0) where $N_5 = 1$, and we get $C_5 = 1$

$$\boxed{N_5 = (1 - \zeta^2)(1 - \eta^2)}$$

Getting the shape functions for 4-quad elements using the same way, and obtaining $C_1 = C_2 = C_3 = C_4 = \frac{1}{4}$ by setting $N_i = 1$ at point of node i.

$$\begin{aligned} N_1' &= C_1 L_{23} L_{34} = \frac{1}{4}(1 - \zeta)(1 - \eta) \\ N_2' &= C_2 L_{34} L_{41} = \frac{1}{4}(1 + \zeta)(1 - \eta) \\ N_3' &= C_3 L_{41} L_{12} = \frac{1}{4}(1 + \zeta)(1 + \eta) \\ N_4' &= C_4 L_{12} L_{23} = \frac{1}{4}(1 - \zeta)(1 + \eta) \end{aligned}$$

For the 5-quad elements

$$\begin{aligned} N_1 &= N_1' + \alpha N_5 = \frac{1}{4}(1 - \zeta)(1 - \eta) + \alpha(1 - \zeta^2)(1 - \eta^2) \\ N_2 &= N_2' + \alpha N_5 = \frac{1}{4}(1 + \zeta)(1 - \eta) + \alpha(1 - \zeta^2)(1 - \eta^2) \\ N_3 &= N_3' + \alpha N_5 = \frac{1}{4}(1 + \zeta)(1 + \eta) + \alpha(1 - \zeta^2)(1 - \eta^2) \\ N_4 &= N_4' + \alpha N_5 = \frac{1}{4}(1 - \zeta)(1 + \eta) + \alpha(1 - \zeta^2)(1 - \eta^2) \end{aligned}$$

To get α, we set the shape functions to zero at node 5 (0,0) obtaining $\alpha = -\frac{1}{4}$

$$\boxed{\begin{aligned} N_1 &= \frac{1}{4}(1 - \zeta)(1 - \eta) - \frac{1}{4}(1 - \zeta^2)(1 - \eta^2) \\ N_2 &= \frac{1}{4}(1 + \zeta)(1 - \eta) - \frac{1}{4}(1 - \zeta^2)(1 - \eta^2) \\ N_3 &= \frac{1}{4}(1 + \zeta)(1 + \eta) - \frac{1}{4}(1 - \zeta^2)(1 - \eta^2) \\ N_4 &= \frac{1}{4}(1 - \zeta)(1 + \eta) - \frac{1}{4}(1 - \zeta^2)(1 - \eta^2) \end{aligned}}$$

The shape functions are compatible since they are polynomials having continuous derivatives, then we verify that the shape functions sum to unity

$$\begin{aligned} N_1 + N_2 + N_3 + N_4 + N_5 &= \frac{1}{4}(1 - \zeta)(1 - \eta) - \frac{1}{4}(1 - \zeta^2)(1 - \eta^2) + \frac{1}{4}(1 + \zeta)(1 - \eta) - \frac{1}{4}(1 - \zeta^2)(1 - \eta^2) \\ &+ \frac{1}{4}(1 + \zeta)(1 + \eta) - \frac{1}{4}(1 - \zeta^2)(1 - \eta^2) + \frac{1}{4}(1 - \zeta)(1 + \eta) - \frac{1}{4}(1 - \zeta^2)(1 - \eta^2) + (1 - \zeta^2)(1 - \eta^2) \\ &= \frac{1}{4}(1 - \zeta - \eta + \zeta\eta) + \frac{1}{4}(1 + \zeta - \eta - \zeta\eta) + \frac{1}{4}(1 + \zeta + \eta + \zeta\eta) + \frac{1}{4}(1 - \zeta + \eta - \zeta\eta) \\ &= 1 \end{aligned}$$

3 Appendix (Developed MATLAB code)

```
clear all;
syms E a b r z g
r1=0; r2=a; r3=a;
z1=0; z2=0; z3=b;

rbar=(r1+r2+r3)/3;
zbar=(z1+z2+z3)/3;

alpha1=r2*z3-r3*z2; alpha2=r3*z1-r1*z3; alpha3=r1*z2-r2*z1;
beta1=z2-z3;        beta2=z3-z1;        beta3=z1-z2;
del1=r3-r2;         del2=r1-r3;         del3=r2-r1;

A=(r1*(z2-z3)+r2*(z3-z1)+r3*(z1-z2))/2;

N1=(alpha1+beta1*r+del1*z)/(2*A);
N2=(alpha2+beta2*r+del2*z)/(2*A);
N3=(alpha3+beta3*r+del3*z)/(2*A);

N=[N1,0,N2,0,N3,0;
    0,N1,0,N2,0,N3];

B=[diff(N1,r),0,diff(N2,r),0,diff(N3,r),0;
    0,diff(N1,z),0,diff(N2,z),0,diff(N3,z);
    N1/r,0,N2/r,0,N3/r,0;
  diff(N1,z),diff(N1,r),diff(N2,z),diff(N2,r),diff(N3,z),diff(N3,r)];

B=subs(B,r,rbar);
B=subs(B,z,zbar);

D=E*[1,0,0,0;
      0,1,0,0;
      0,0,1,0;
      0,0,0,0.5];

K=2*pi*A*rbar*transpose(B)*D*B;
simplify(K)

N=subs(N,r,rbar);
N=subs(N,z,zbar);

b=[0;-g];
F=(2*pi*A*rbar)*transpose(N)*b
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