Universitat Politecnica de Catalunya CIMNE
Master in Computational Mechanics

Computational Structural Mechanics & Dynamics

Assignment 4
Isoparametric Representation & Structures of Revolution

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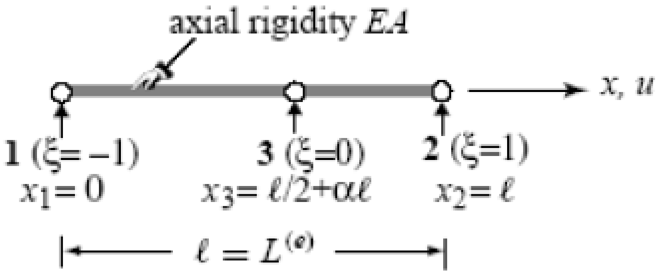
Assignment 4.1 :



$x_1 = 0\,, x_2 = l\,, x_3 = \left(\frac{1}{2} + \alpha\right) l$ and $\xi_1 = 1\,, \xi_2 = -1\,, \xi_3 = 0$

Solution:

1)

The Shape functions for 1D bar element with 3 nodes are:

$$N_1 = \frac{\xi(\xi-1)}{2}\,, \quad N_2 = \frac{\xi(\xi+1)}{2}, \quad N_3 = 1 - \xi^2$$

$$x = N_1(\xi)x_1 + N_2(\xi)x_2 + N_3(\xi)x_3$$

So,

$$J = \frac{dx}{d\xi} = \frac{dN_1}{d\xi}x_1 + \frac{dN_2}{d\xi}x_2 + \frac{dN_3}{d\xi}x_3$$

$$= 0 + \left(\frac{1}{2} + \xi\right) l - 2\xi(\frac{1}{2} + \alpha)l$$

$$= 0 + \left(\frac{1}{2} + \xi\right) l - 2\xi(\frac{1}{2} + \alpha)l$$

$$= \left(\frac{l}{2}\right) + \xi l - \xi l - 2\xi\alpha l$$

$$J = \left(\frac{1}{2} - 2\xi\alpha\right) l$$

- In order to show that if $\frac{-1}{4} < \alpha < \frac{1}{4}$ then J > 0 over the whole element $-1 < \xi < 1$,

  We substitute the limits in the above equation of J, which gives us,

  For $\alpha = \frac{-1}{4}$: $J = \left(\frac{1}{2} - 2\xi(\frac{-1}{4})\right) l = (1 + \xi)\frac{l}{2}$

  For $\alpha = \frac{1}{4}$: $J = \left(\frac{1}{2} - 2\xi(\frac{1}{4})\right) l = (1 - \xi)\frac{l}{2}$

  As in both the above cases we observe that, the value of J always be greater than zero.

- And, in the other case when $\alpha = 0$, the value of J is $\frac{l}{2}$

  $\therefore J = \left(\frac{1}{2} - 2\xi(0)\right) l = \frac{l}{2}$

2)

The strain displacement matrix is given by;

$$B = \frac{dN}{dx} = J^{-1}\frac{dN}{d\xi}$$

$$B = \frac{1}{\left(\frac{1}{2}-2\xi\alpha\right)l}\begin{bmatrix}\frac{dN_1}{d\xi} & \frac{dN_2}{d\xi} & \frac{dN_3}{d\xi}\end{bmatrix}$$

$$\boxed{B = \frac{1}{\left(\frac{1}{2}-2\xi\alpha\right)l}\left[\left(\xi-\frac{1}{2}\right), \quad \left(\xi+\frac{1}{2}\right), \quad -2\xi\right]}$$

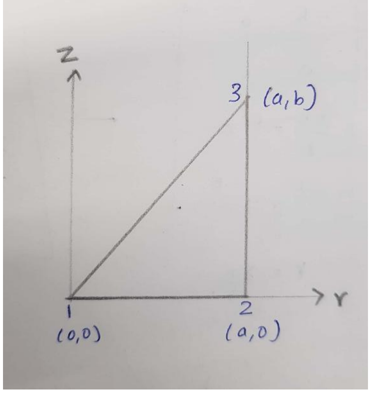
Assignment 4.2 :

$$r_1 = 0\,, \;\; r_2 = r_3 = a\,, \;\; z_1 = z_2 = 0\,, \;\; z_3 = b$$

$$\mathbf{E} = E\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & \frac{1}{2}\end{bmatrix}, \qquad \mathrm{b} = \begin{bmatrix}0\\ -g\end{bmatrix}$$

Solution:

1)



The stiffness K^e for the given plane stress triangle is given by:

$$K^e = 2\pi B^T E B \bar{r} A$$

Where,

$$B_i = \begin{bmatrix}\frac{\partial N_i}{\partial r} & 0\\ 0 & \frac{\partial N_i}{\partial z}\\ \frac{N_i}{r} & 0\\ \frac{\partial N_i}{\partial z} & \frac{\partial N_i}{\partial r}\end{bmatrix}$$

$$N_i = \frac{a_i + b_i r + c_i z}{2\Delta}$$

$$a_i = r_j z_m + r_m z_j\,, \qquad b_i = z_j - z_m\,, \qquad c_i = r_m - r_j$$

$$\bar{r} = \frac{r_i + r_j + r_m}{3} = \frac{2a}{3}\,, \quad \bar{z} = \frac{z_i + z_j + z_m}{3} = \frac{b}{3}$$

The area of triangle is given by,

$$\Delta = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ r_1 & r_2 & r_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

$$\Delta = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 0 & a & a \\ 0 & 0 & b \end{bmatrix}$$

$$\Delta = \frac{ab}{2}$$

$a_1 = ab$, $a_2 = 0$, $a_3 = 0$

$b_1 = -b$, $b_2 = b$, $b_3 = 0$

$c_1 = 0$, $c_2 = -a$, $c_3 = 0$

So,

$$N_1 = \frac{a_1 + b_1 r + c_1 z}{2 * \frac{ab}{2}} = 1 - \frac{r}{a} \qquad \frac{\partial N_1}{\partial r} = -\frac{1}{a} \qquad \frac{\partial N_1}{\partial z} = 0$$

$$N_2 = \frac{a_2 + b_2 r + c_2 z}{2 * \frac{ab}{2}} = \frac{r}{a} - \frac{z}{b} \qquad \frac{\partial N_2}{\partial r} = \frac{1}{a} \qquad \frac{\partial N_2}{\partial z} = -\frac{1}{b}$$

$$N_3 = \frac{a_3 + b_3 r + c_3 z}{2 * \frac{ab}{2}} = \frac{z}{b} \qquad \frac{\partial N_3}{\partial r} = 0 \qquad \frac{\partial N_3}{\partial z} = \frac{1}{b}$$

The B matrix becomes, $$B = \begin{bmatrix} \frac{\partial N_1}{\partial r} & 0 & \frac{\partial N_2}{\partial r} & 0 & \frac{\partial N_3}{\partial r} & 0 \\ 0 & \frac{\partial N_1}{\partial z} & 0 & \frac{\partial N_2}{\partial z} & 0 & \frac{\partial N_3}{\partial z} \\ \frac{N_1}{r} & 0 & \frac{N_2}{r} & 0 & \frac{N_3}{r} & 0 \\ \frac{\partial N_1}{\partial z} & \frac{\partial N_1}{\partial r} & \frac{\partial N_2}{\partial z} & \frac{\partial N_2}{\partial r} & \frac{\partial N_3}{\partial z} & \frac{\partial N_3}{\partial r} \end{bmatrix}$$

$$B = \begin{bmatrix} -\frac{1}{a} & 0 & \frac{1}{a} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{b} & 0 & \frac{1}{b} \\ \frac{1}{r} - \frac{1}{a} & 0 & \frac{1}{a} - \frac{z}{rb} & 0 & \frac{z}{rb} & 0 \\ 0 & -\frac{1}{a} & -\frac{1}{b} & \frac{1}{a} & \frac{1}{b} & 0 \end{bmatrix}$$

Evaluating this matrix at the centroidal point $(\frac{2a}{3}, \frac{b}{3})$

$$B = \begin{bmatrix} -\frac{1}{a} & 0 & \frac{1}{a} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{b} & 0 & \frac{1}{b} \\ \frac{1}{2a} & 0 & \frac{1}{2a} & 0 & \frac{1}{2a} & 0 \\ 0 & -\frac{1}{a} & -\frac{1}{b} & \frac{1}{a} & \frac{1}{b} & 0 \end{bmatrix}$$

The stiffness matrix then becomes,

$$K = 2\pi B^T E B \bar{r} A$$

$$K = 2\pi B^T E B * \frac{2a}{3} * \frac{ab}{2}$$

$$K = \frac{2\pi a^2 b}{3} \begin{bmatrix} -\frac{1}{a} & 0 & \frac{1}{2a} & 0 \\ 0 & 0 & 0 & -\frac{1}{a} \\ \frac{1}{a} & 0 & \frac{1}{2a} & -\frac{1}{b} \\ 0 & -\frac{1}{b} & 0 & \frac{1}{a} \\ 0 & 0 & \frac{1}{2a} & \frac{1}{b} \\ 0 & \frac{1}{b} & 0 & 0 \end{bmatrix} E \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -\frac{1}{a} & 0 & \frac{1}{a} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{b} & 0 & \frac{1}{b} \\ \frac{1}{2a} & 0 & \frac{1}{2a} & 0 & \frac{1}{2a} & 0 \\ 0 & -\frac{1}{a} & -\frac{1}{b} & \frac{1}{a} & \frac{1}{b} & 0 \end{bmatrix}$$

$$\boxed{K = \pi E \begin{bmatrix} \frac{5b}{6} & 0 & -\frac{b}{2} & 0 & \frac{b}{6} & 0 \\ 0 & \frac{b}{3} & \frac{a}{3} & -\frac{b}{3} & -\frac{a}{3} & 0 \\ -\frac{b}{2} & \frac{a}{3} & \frac{2a^2+5b^2}{6b} & -\frac{a}{3} & \frac{-2a^2+b^2}{6b} & 0 \\ 0 & -\frac{b}{3} & -\frac{a}{3} & \frac{2a^2+b^2}{3b} & \frac{a}{3} & \frac{-2a^2}{3b} \\ \frac{b}{6} & -\frac{a}{3} & \frac{-2a^2+b^2}{6b} & \frac{a}{3} & \frac{2a^2+b^2}{6b} & 0 \\ 0 & 0 & 0 & \frac{-2a^2}{3b} & 0 & \frac{2a^2}{3b} \end{bmatrix}}$$

2)

From the above matrix, we can make 2 observations as, first, the sum of rows and columns 1, 3 & 5 does not become 0, while the sum of rows and columns 2, 4 & 6 is balanced and becomes 0. The odd number of rows and columns represent r – direction of the structure and the even rows and columns represent the z – direction of structure.

The summation of columns 1, 3 & 5 is shown below which does not vanish.

$$\begin{bmatrix} \frac{5b}{6} \\ 0 \\ -\frac{b}{2} \\ 0 \\ \frac{b}{6} \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{b}{2} \\ \frac{a}{3} \\ \frac{2a^2+5b^2}{6b} \\ -\frac{a}{3} \\ \frac{-2a^2+b^2}{6b} \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{b}{6} \\ -\frac{a}{3} \\ \frac{-2a^2+b^2}{6b} \\ \frac{a}{3} \\ \frac{2a^2+b^2}{6b} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{b}{2} \\ 0 \\ \frac{b}{2} \\ 0 \\ \frac{b}{2} \\ 0 \end{bmatrix}$$

The summation of columns 2, 4 & 6 is shown below which vanishes.

$$\begin{bmatrix} 0 \\ \frac{b}{3} \\ \frac{a}{3} \\ -\frac{b}{3} \\ -\frac{a}{3} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{b}{3} \\ -\frac{a}{3} \\ \frac{2a^2+b^2}{3b} \\ \frac{a}{3} \\ \frac{-2a^2}{3b} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{-2a^2}{3b} \\ 0 \\ \frac{2a^2}{3b} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The same can be observed for the summation of the rows.

The sum of rows and columns becoming 0 implies that rigid body motion is possible in that direction. Whereas, if the summation of the rows and columns is not zero, it indicates that there would be no rigid body motion in that particular direction in which the sum of rows and columns is not balanced.

3)

The consistent force vector for gravity forces is computed as,

$$f^e = \int N^T br\, dA$$

$$f^e = \frac{r * ab}{2} \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \end{bmatrix} \begin{bmatrix} 0 \\ -g \end{bmatrix}$$

Calculating at the centroidal point,

$$f^e = \frac{a^2 b}{3} \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \\ \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \\ \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 \\ -g \end{bmatrix}$$

$$\boxed{\therefore\ f^e = \frac{-a^2 bg}{9} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}}$$

Discussions:

- In the first task, isoparametric representation of 1D bar element with 3 nodes was studied which involved finding the Jacobian and the strain displacement matrix. The main idea of the Jacobian is that, it relates the natural co-ordinates of a geometry with the computational co-ordinates.
- In the second task, the stiffness and consistent force vector of an axis-symmetric triangle were calculated using centroidal co-ordinates to ease off the calculation complexities. The structures of revolution are rotationally symmetric and stresses and strains are independent of the circumferential co-ordinates. In this case, it was observed that the summation of rows and columns in direction of symmetry axis is zero thus allowing rigid body motion in that direction. While, summation of rows and columns along the radial direction was unbalanced due to stresses and strains acting in that direction and hence did not reduce to zero.