INTERNATIONAL CENTRE FOR
NUMERICAL METHODS IN ENGINEERING

UNIVERSITAT POLITÈCNICA DE CATALUNYA

MASTER OF SCIENCE IN COMPUTATIONAL MECHANICS

Computational Structural Mechanics and Dynamics

Assignment 4

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ASSIGNMENT 4.1

***1.** Compute the entries of* $\mathbf{K}^e$ *for the following axisymmetric triangle:*

$$r_1 = 0, \quad r_2 = r_3 = a, \quad z_1 = z_2 = 0, \quad z_3 = b$$

The material is isotropic with $\nu = 0$ for which stress-strain matrix is,

$$E \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix} \tag{0.1}$$

Solution 1:

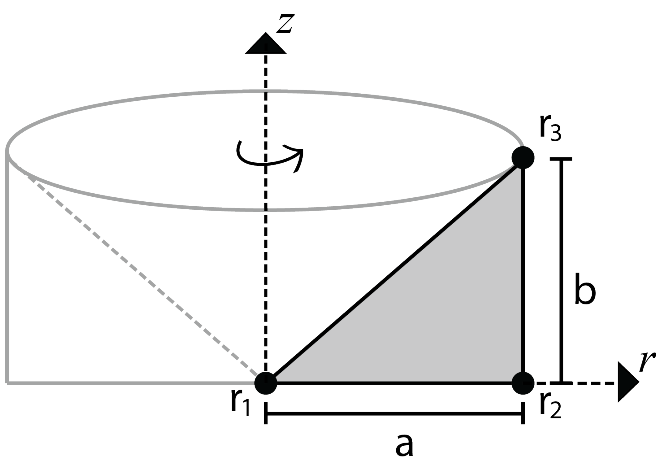

Figure 0.1: Discretization of one axisymmetric triangle

The element stiffness matrix of an axisymmetric linear triangular element is defined as:

$$\mathbf{K}^e = 2\pi \int_A r\mathbf{B}^T\mathbf{E}\mathbf{B}dA \tag{0.2}$$

In which it should be noticed that the term 2π can be neglected when it is time to solve the system of equations, due to the same term that multiplies the force vector. But as a reminder that the expression 0.2 is a solid of revolution of 360º respect to a central axis, the term will be taken as constant. Then, $\mathbf{E}$ is defined as in the equation 0.1, and $\mathbf{B}_i$ is defined as the partial derivatives of the shape function respect to each coordinate:

$$\mathbf{B}_i = \frac{1}{2A^{(e)}} \begin{bmatrix} \frac{\partial N_i}{\partial r} & 0 \\ 0 & \frac{\partial N_i}{\partial z} \\ \frac{N_i}{r} & 0 \\ \frac{\partial N_i}{\partial z} & \frac{\partial N_i}{\partial r} \end{bmatrix} \tag{0.3}$$

The shape functions of the axisymmetric triangle are defined as:

$$N_i = \frac{1}{2A^{(e)}}(a_i + b_i r + c_i z) \tag{0.4}$$

where:

$$\begin{aligned} a_i &= r_j z_k - r_k z_j \\ b_i &= z_j - z_k \\ c_i &= r_k - r_j \end{aligned}$$

Now, in order to calculate each shape function and the components of the matrix $\mathbf{B}_i$ the table 0.1 is considered, then:

Nodes	r	z	a_i	b_i	c_i
1	0	0	ab	-b	0
2	a	0	0	b	-a
3	a	b	0	0	a

Table 0.1: Components needed to calculate the shape functions and its derivatives.

$$\begin{aligned} N_1 &= \frac{1}{2A}(ab - br) = 1 - \frac{r}{a} \\ N_2 &= \frac{1}{2A}(br - az) = \frac{r}{a} - \frac{z}{b} \\ N_3 &= \frac{1}{2A}(az) = \frac{z}{b} \end{aligned}$$

Where the area of the triangle is $A = \frac{ab}{2}$. Therefore, the matrix $\mathbf{B}$ is:

$$\mathbf{B}_i = \left[\begin{array}{cc|cc|cc} -\frac{1}{a} & 0 & \frac{1}{a} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{b} & 0 & \frac{1}{b} \\ \frac{a-r}{ar} & 0 & \frac{br-az}{abr} & 0 & \frac{z}{br} & 0 \\ 0 & -\frac{1}{a} & -\frac{1}{b} & \frac{1}{a} & \frac{1}{b} & 0 \end{array}\right] \tag{0.5}$$

Substituting in the stiffness matrix of the axisymmetric triangle and performing the corresponding matrix multiplications:

$$\mathbf{K}^e = \int_A 2\pi E \left[\begin{matrix} -\frac{2}{a}+\frac{2r}{a^2}+\frac{1}{r} & 0 & \frac{z}{ab}+\frac{1}{a}-\frac{2r}{a^2}-\frac{z}{br} \\ 0 & \frac{r}{2^2} & \frac{r}{2ab} \\ \frac{z}{ab}+\frac{1}{a}-\frac{2r}{a^2}-\frac{z}{br} & \frac{r}{2ab} & \frac{-2z}{ab}+\frac{2r}{a^2}+\frac{r}{2b^2}+\frac{z^2}{rb^2} \\ 0 & -\frac{r}{2a^2} & -\frac{r}{2ab} \\ \frac{z}{br}-\frac{z}{ab} & -\frac{r}{2ab} & \frac{z}{ab}-\frac{r}{2b^2}-\frac{z^2}{rb^2} \\ 0 & 0 & 0 \end{matrix}\right. \cdots$$

$$\cdots \left.\begin{matrix} 0 & \frac{z}{br}-\frac{z}{ab} & 0 \\ -\frac{r}{2a^2} & -\frac{r}{2ab} & 0 \\ -\frac{r}{2ab} & \frac{z}{ab}-\frac{r}{2b^2}-\frac{z^2}{rb^2} & 0 \\ \frac{r}{2a^2}+\frac{r}{b^2} & \frac{r}{2ab} & -\frac{r}{b^2} \\ \frac{r}{2ab} & \frac{r}{2b^2}+\frac{z^2}{rb^2} & 0 \\ -\frac{r}{b^2} & 0 & \frac{r}{b^2} \end{matrix}\right] dA$$

In order to integrate the above stiffness matrix, the proposed procedure is by integrating numerically by using Gauss quadratures, first it it needed to transform the area integral into two integrals with normalized limits, in that sense the first calculation required is the Jacobian matrix, which come out by considering the following coordinates transformation:

$$\begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} = \begin{bmatrix} \xi \\ \eta \\ 1-\xi-\eta \end{bmatrix} \tag{0.6}$$

The linear approximation considered as:

$$\begin{aligned} r &= N_1 r_1 + N_2 r_2 + N_3 r_3 \\ z &= N_1 z_1 + N_2 z_2 + N_3 z_3 \end{aligned} \tag{0.7}$$

And is transformed with the natural coordinates as:

$$\begin{aligned} r &= (r_1 - r_3)\xi + (r_2 - r_3)\eta + r_3 \\ z &= (z_1 - z_3)\xi + (z_2 - z_3)\eta + z_3 \end{aligned} \tag{0.8}$$

Performing the derivation of the shape functions N_i with respect to the natural coordenates ξ and η:

$$\begin{bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial r}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial r}{\partial \eta} & \frac{\partial z}{\partial \eta} \end{bmatrix} \begin{bmatrix} \frac{\partial N_i}{\partial r} \\ \frac{\partial N_i}{\partial z} \end{bmatrix} \tag{0.9}$$

Where the Jacobian is:

$$\begin{bmatrix} \frac{\partial r}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial r}{\partial \eta} & \frac{\partial z}{\partial \eta} \end{bmatrix} = \begin{bmatrix} -a & -b \\ 0 & -b \end{bmatrix} \tag{0.10}$$

And its determinant is equal to $det(\mathbf{J}) = ab$. Then, by using the expressions explained above, the Jacobian can be related as the parameter of changing the system of coordinates from cartesian to natural as $drdz = det(\mathbf{J})d\xi d\eta$. The weights (that will be considered as one in the numerical integration) are multiplied by 1/2 so that the element area is correctly computed in those cases.

$$d\Omega = dxdy = det(\mathbf{J})d\xi d\eta \rightarrow \mathbf{K}^e = 2\pi \int_A r\mathbf{B}^T\mathbf{E}\mathbf{B}\frac{det(\mathbf{J})}{2}d\xi d\eta \tag{0.11}$$

The general approximate solution can be obtained as:

$$2\pi \int_A r\mathbf{B}^T\mathbf{E}\mathbf{B}\frac{det(\mathbf{J})}{2}d\xi d\eta \approx 2\pi \sum_{i=1}^{p1}\sum_{j=1}^{p2} w_i w_j r\mathbf{B}^T(\xi_i,\eta_j)\mathbf{E}\mathbf{B}(\xi_i,\eta_j)\frac{det(\mathbf{J})}{2} \tag{0.12}$$

The Gauss Quadrature chosen is one point of integration located in the centroid ($r = 2a/3$ and $z = b/3$) of the triangular element and both weights are equal to 1. Integrating numerically, the stiffness matrix (neglecting the $2\pi parameter$ becomes :

$$\mathbf{K}^e = \frac{E}{2}\int \begin{bmatrix} \frac{5b}{3} & 0 & -b & 0 & \frac{b}{3} & 0 \\ 0 & \frac{2b}{3} & \frac{2a}{3} & -\frac{2b}{3} & -\frac{2a}{3} & 0 \\ -b & \frac{2a}{3} & \frac{2a^2}{3b}+\frac{5b}{3} & -\frac{2a}{3} & \frac{b}{3}-\frac{2a^2}{3b} & 0 \\ 0 & -\frac{2b}{3} & -\frac{2a}{3} & \frac{4a^2}{3b}+\frac{2b}{3} & \frac{2a}{3} & -\frac{4a^2}{3b} \\ \frac{b}{3} & -\frac{2a}{3} & \frac{b}{3}-\frac{2a^2}{3b} & \frac{2a}{3} & \frac{2a^2}{3b}+\frac{b}{3} & 0 \\ 0 & 0 & 0 & -\frac{4a^2}{3b} & 0 & \frac{4a^2}{3b} \end{bmatrix} dA$$

2. *Show that the sum of the rows (and columns) 2, 4 and 6 of* $\mathbf{K}^e$ *must vanish and explain why. Show as well that the sum of rows (and columns) 1, 3 and 5 does not vanish, and explain why.*

Solution 2:

Row and column 2:

$$\frac{2b}{3}+\frac{2a}{3}-\frac{2b}{3}-\frac{2a}{3}=0$$

Row and column 4:

$$-\frac{2b}{3}-\frac{2a}{3}+\frac{4a^2}{3b}+\frac{2b}{3}+\frac{2a}{3}-\frac{4a^2}{3b}=0$$

Row and column 6:

$$\frac{b}{3}-\frac{2a}{3}+\frac{b}{3}-\frac{2a^2}{3b}-\frac{4a^2}{3b}+\frac{4a^2}{3b}=0$$

The explanation of why the sum of the components related to the 2, 4 and 6 columns and rows are equal to zero, is because at this point the formulation of the stiffness matrix is unconstrained, the nodes are free to move and due to that the rigidity of those degrees of freedom are equal to zero, allowing to have a rigid body motion. In that sense, the next step to perform a complete analysis is to add restrictions in required nodes.

Row and column 1:

$$\frac{5b}{3}-b+\frac{b}{3}=b$$

Row and column 3:

$$-b+\frac{2a}{3}+\frac{2a^2}{3b}+\frac{5b}{3}-\frac{2a}{3}+\frac{b}{3}-\frac{2a^2}{3b}=b$$

Row and column 5:

$$\frac{b}{3}-\frac{2a}{3}+\frac{b}{3}-\frac{2a^2}{3b}+\frac{2a}{3}+\frac{2a^2}{3b}+\frac{b}{3}=b$$

In contrast with the previous example, which the rows are related to the z direction components and their sum is equal to zero, in this second demonstration the rows and columns 1, 3 and 5 correspond to the degrees of freedom in the r direction and the result of the sum is different to zero. This is because of the intrinsic restriction in the matrix due to the axisymmetric condition, in other words, the nodes are not permitted to move completely free through this axis, because of the stiffness imposed.

3. *Compute the consistent force vector fe for gravity forces* $b = [0, -g]^T T$.

Solution 3:

The force vector for body forces is given by the expression:

$$\mathbf{f}_b = \int_A r\mathbf{N}\mathbf{b}dA \tag{0.13}$$

In which it is written the value of 2π just to remember that this vector is part of a structure of revolution, but this value will cancel at the time of solving the system of equations. Now, substituting the shape functions and the body force applied in the z direction (neglecting the 2π parameter):

$$\mathbf{f}_b = \int_A -g \begin{bmatrix} 0 \\ r - \frac{r^2}{a} \\ 0 \\ \frac{r^2}{a} - \frac{zr}{b} \\ 0 \\ \frac{zr}{b} \end{bmatrix} dA$$

Integrating numerically the force vector, by using the same approach employed in the stiffness matrix:

$$\mathbf{f}_b = \int_A \mathbf{N}\mathbf{b}\frac{det(\mathbf{J})}{2} d\xi d\eta \approx \sum_{i=1}^{p1} w_i \mathbf{N}\mathbf{b}\frac{det(\mathbf{J})}{2}$$

Then, the force vector is:

$$\mathbf{f}_b = -g \begin{bmatrix} 0 \\ \frac{a^2 b}{9} \\ 0 \\ \frac{a^2 b}{9} \\ 0 \\ \frac{a^2 b}{9} \end{bmatrix} dA$$

Assignment 4.2

A five node quadrilateral element has the nodal configuration shown if the figure. Perspective views of N_1^e and N_5^e are shown in the same figure.
Find five shape functions N_i^e, $i = 1, \ldots, 5$ that satisfy compatibility and also verify that their sum is unity.

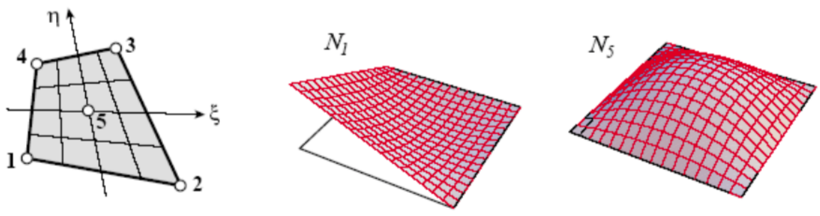

Figure 0.2: Five node quadrilateral element

Hint: develop $N_5(\xi,\eta)$ first for the 5-node quad using the line-product method. Then the corner shape functions $\underline{N_i}(\xi,\eta)$, $i = 1,2,3,4$, for the 4-node quad (already given in the notes). Finally combine $N_i = \underline{N_i} + \alpha N_5$ determining α so that all N_i vanish node 5. Check that $N_1 + N_2 + N_3 + N_4 + N_5 = 1$ identically.

Solution:

The method required to use in order to obtain the shape functions works as a direct consequence of the definition of the natural coordinates, and it is based on simple polynomial products. It is known that the method itself requires some rules to be verified. In that sense not every element can be generated by this method, for instance the 5-node quadrilateral is one specific case which it can not obtained directly using this method only. To perform this procedure, first it required to obtain the line product of N_5:

$$N_5 = c_5 L_{1-2} L_{2-3} L_{3-4} L_{4-1}$$

For this shape function, we can observe that the equations of each side are:

- Side 1-2: $\eta = -1$
- Side 2-3: $\xi = 1$
- Side 3-4: $\eta = 1$
- Side 4-1: $\xi = -1$

Considering these equations, the expression for the node 5 is:

$$N_5 = c_5(1+\eta)(1-\xi)(1-\eta)(1+\xi) \tag{0.14}$$

Which vanishes over each node (1,2, 3 and 4). Now to obtain the value of c_5, substitute the values of the natural coordinate $\xi = 0$ and $\eta = 0$

$$N_5 = c_5(1+0)(1-0)(1-0)(1+0) = 1$$

Then $c_5 = 1$, and the partial shape function is:

$$N_5(\xi,\eta) = (1+\eta)(1-\xi)(1-\eta)(1+\xi) = (1-\xi^2)(1-\eta^2) \tag{0.15}$$

Now, as the hint mentions, it is needed to obtain a relationship between the above equation and the rest of the functions of the nodes 1, 2, 3 and 4 ($\underline{N_i}$), by using a parameter α.

$$N_i = \underline{N_i} + \alpha N_5$$

Considering the first node N_1:

$$\begin{aligned} N_1 &= \underline{N_1} + \alpha N_5 \\ &= \frac{1}{4}(1-\xi)(1-\eta) + \alpha(1-\xi^2)(1-\eta^2) \end{aligned}$$

Substituting the natural coordinates of the node 5:

$$\frac{1}{4}(1-0)(1-0) + \alpha(1-0^2)(1-0^2) = 0$$

Solving for the value α:

$$\alpha = -\frac{1}{4} \tag{0.16}$$

Substituting this value in each shape function of the nodes 1, 2,3 and 4:

$$\begin{aligned} N_1 &= \frac{1}{4}(1-\xi)(1-\eta) - \frac{1}{4}(1-\xi^2)(1-\eta^2) \\ &= \frac{1}{4}(1-\xi)(1-\eta)[1-(1+\xi)(1+\eta)] \\ &= \frac{1}{4}(1-\xi)(1-\eta)(-\xi-\eta-\xi\eta) \end{aligned}$$

Node 1:

$$N_1 = -\frac{1}{4}(1-\xi)(1-\eta)(\xi+\eta+\xi\eta) \tag{0.17}$$

$$
\begin{aligned}
N_2 &= \underline{N_2} - \frac{1}{4} N_5 \\
&= \frac{1}{4}(1+\xi)(1-\eta) - \frac{1}{4}(1-\xi^2)(1-\eta^2) \\
&= \frac{1}{4}(1+\xi)(1-\eta)[1-(1-\xi)(1+\eta)]
\end{aligned}
$$

Node 2:

$$
N_2 = \frac{1}{4}(1+\xi)(1-\eta)(\xi-\eta+\xi\eta) \tag{0.18}
$$

$$
\begin{aligned}
N_3 &= \underline{N_3} - \frac{1}{4} N_5 \\
&= \frac{1}{4}(1+\xi)(1+\eta) - \frac{1}{4}(1-\xi^2)(1-\eta^2) \\
&= \frac{1}{4}(1+\xi)(1+\eta)[1-(1-\xi)(1-\eta)]
\end{aligned}
$$

Node 3:

$$
N_3 = \frac{1}{4}(1+\xi)(1+\eta)(\xi+\eta-\xi\eta) \tag{0.19}
$$

$$
\begin{aligned}
N_4 &= \underline{N_4} - \frac{1}{4} N_5 \\
&= \frac{1}{4}(1-\xi)(1+\eta) - \frac{1}{4}(1-\xi^2)(1-\eta^2) \\
&= \frac{1}{4}(1-\xi)(1+\eta)[1-(1+\xi)(1-\eta)]
\end{aligned}
$$

Node 4:

$$
N_4 = \frac{1}{4}(1-\xi)(1+\eta)(-\xi+\eta+\xi\eta) \tag{0.20}
$$

In order to prove the compatibility of the shape functions, it is required that one node of the interelement boundaries is analyzed, to perform this, it is necessary to verify the order of each polynomial and test that for “n” order, there are “n+1” points. The function of node 1 is tested and generalized for the rest of the nodes, this node belongs to the sides 1-2 and 1-4.

Side 1-2 ($\eta = -1$):

$$\begin{aligned} N_1 &= \frac{1}{4}(1-\xi)(1-(-1))(-\xi-(-1)-\xi(-1)) \\ &= \frac{1}{2}(1-\xi) \end{aligned}$$

Side 1-4 ($\xi = -1$):

$$\begin{aligned} N_1 &= \frac{1}{4}(1-(-1))(1-\eta)(-(-1)-\eta-(-1)\eta) \\ &= \frac{1}{2}(1-\eta) \end{aligned}$$

Both shape functions are linear, that means $n = 1$ and have two nodes, $n+1 = 2$, so the compatibility condition is verified. This analysis is identical to the functions 2, 3 and 4.
Now, to verify that the sum of the shape functions is equal to 1, the first 4 equations are considered and evaluated:

Node 1 ($\xi = -1, \eta = -1$):

$$\begin{aligned} N_1 &= -\frac{1}{4}(1-\xi)(1-\eta)(\xi+\eta+\xi\eta) \\ &= -\frac{1}{4}(1-(-1))(1-(-1))(-1-1+(-1)(-1) \\ &= 1 \end{aligned}$$

Node 2($\xi = 1, \eta = -1$):

$$\begin{aligned} N_2 &= \frac{1}{4}(1+\xi)(1-\eta)(\xi-\eta+\xi\eta) \\ &= \frac{1}{4}(1+(-1))(1-(-1))(-1-1+(-1)(-1) \\ &= 1 \end{aligned}$$

Node 3($\xi = 1, \eta = 1$):

$$\begin{aligned} N_3 &= \frac{1}{4}(1+\xi)(1+\eta)(\xi+\eta-\xi\eta) \\ &= \frac{1}{4}(1+(-1))(1+(-1))(-1-1+(-1)(-1) \\ &= 1 \end{aligned}$$

Node 4($\xi = -1, \eta = 1$):

$$
\begin{aligned}
N_4 &= \frac{1}{4}(1-\xi)(1+\eta)(-\xi+\eta+\xi\eta) \\
&= \frac{1}{4}(1-(-1))(1+(-1))(-1+1+(-1)(-1) \\
&= 1
\end{aligned}
$$

The fifth function was previously verified in equation 0.15.