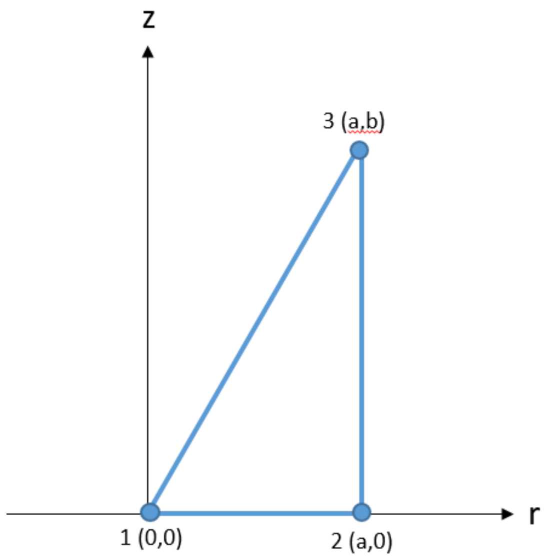
Computational Structural Mechanics and Dynamics

Assignment 4 – Trond Jorgen Opheim



1. **Compute the entries of K for the axisymmetric triangle**

To compute the stiffness matrix for the element given in Figure 1, I use the properties from polar coordinates, which gives me the following expression

$$K = \int_V \boldsymbol{B^T E B} dv$$

$$K = \int_0^{2\pi} \int_A \boldsymbol{B^T E B} dA d\theta$$

$$K = 2\pi \int_A \boldsymbol{B^T E B} dA$$

To find the B-matrix I start by finding the shape functions, and then using the following properties

$$B = \frac{1}{2A} \begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 z}{r} & 0 & \frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 z}{r} & 0 & \frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r} & 0 \\ \gamma_1 & \beta_2 & \gamma_2 & \beta_3 & \gamma_3 & \beta_4 \end{bmatrix}$$

$$u = \frac{1}{2A} [1 \;\; r \;\; z] \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

Shape functions N_1, N_2, N_3

$$u(r,z) = N_1 u_1 + N_2 u_2 + N_3 U_3$$

$$u(r,z) = C_1 + C_2 r + C_3 z$$

$$u(0{,}0) = u_1 \qquad u(a,0) = u_2 \qquad u(a,b) = u_3$$

$$u(r,z) = \left(1 - \frac{r}{a}\right) u_1 + \left(\frac{r}{a} - \frac{z}{b}\right) u_2 + \left(\frac{z}{b}\right) u_3$$

$$u(r,z) = \frac{1}{ab} [1 \;\; r \;\; z] \begin{bmatrix} ab & 0 & 0 \\ -b & b & 0 \\ 0 & -a & a \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

By transforming this equation into a matrix equation on the form as (1.1), I obtain the values for α, β, γ and the B-matrix for the triangular element becomes the following

$$B = \frac{1}{ab} \begin{bmatrix} -b & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & -a & 0 & a \\ \frac{ab}{r} - b & 0 & b - \frac{az}{r} & 0 & \frac{az}{r} & 0 \\ 0 & -b & -a & b & a & 0 \end{bmatrix}$$

Calculation of K

Since the integrand is dependent of both the variables in the integral, the whole equation gets very complicated. To calculate the stiffness matrix I therefore use an approximation given by

$$K = 2\pi \bar{r} \overline{\boldsymbol{B}^{\boldsymbol{T}}} \boldsymbol{E} \overline{\boldsymbol{B}}$$

$$\bar{r} = \frac{r_1 + r_2 + r_3}{3} \qquad \bar{z} = \frac{z_1 + z_2 + z_3}{3} \qquad \overline{\boldsymbol{B}} = \boldsymbol{B}(\bar{r}, \bar{z})$$

By inserting $r = \bar{r}$ and $z = \bar{z}$, and doing the calculations in Matlab I get the following stiffness matrix

$$K = E\pi \begin{bmatrix} \frac{b}{6} & 0 & -\frac{b}{2} & 0 & \frac{b}{6} & 0 \\ 0 & \frac{b}{3} & \frac{a}{3} & -\frac{b}{3} & -\frac{a}{3} & 0 \\ -\frac{b}{2} & \frac{a}{3} & \frac{a^2}{3b} + \frac{5b}{6} & -\frac{a}{3} & -\frac{a^2}{3b} + \frac{b}{6} & 0 \\ 0 & -\frac{b}{3} & -\frac{a}{3} & \frac{2a^2}{3b} + \frac{b}{3} & \frac{a}{3} & -\frac{2a^2}{3b} \\ \frac{b}{6} & -\frac{a}{3} & -\frac{a^2}{3b} + \frac{b}{6} & \frac{a}{3} & \frac{a^2}{3b} + \frac{b}{6} & 0 \\ 0 & 0 & 0 & -\frac{2a^2}{3b} & 0 & \frac{2a^2}{3b} \end{bmatrix}$$

2. Sum of rows and columns

As one can see from the stiffness matrix, if one sums up the entries in row or column 2, 4 and 6, the sum is equal to zero. This situation describes a rigid body motion of the solid figure, which the solid figure has no stiffness or forces to prevent. The basic formulation in finite element is $K * r = f$, and if $r = 1$ and there are no external forces acting on it, the sum of forces have to be equal zero which is handled by the summation of entries in K.

For the case row and column 1, 3 and 5 we are looking at the situation where the solid is expanding in the r-direction. Since the solid figure in this situation obviously is being stretched, the solid figure has a stiffness to prevent this displacement.

3. Consistent force vector

The consistent force vector is given by equation (3.1). By using the same approximation as in problem 1 and with the computed shape functions, I get the following results

$$f = 2\pi \int_A \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \end{bmatrix} \begin{bmatrix} 0 \\ -g \end{bmatrix} \bar{r} dr dz$$

$$N_1 = 1 - \frac{\bar{r}}{a} \qquad N_2 = \frac{\bar{r}}{a} - \frac{\bar{z}}{b} \qquad N_3 = \frac{\bar{z}}{b}$$

$$f_i = \frac{2\pi}{3} A\bar{r} \begin{bmatrix} 0 \\ -g \end{bmatrix}$$

$$f = -\frac{2\pi a^2 b}{9} \begin{bmatrix} 0 \\ g \\ 0 \\ g \\ 0 \\ g \end{bmatrix}$$