Assignment 4

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March 9, 2020

Assignment 4.1

1) This is a 3 noded 1D element, therefore the shape functions are the following:

$$N_1 = \frac{\xi(\xi-1)}{2}$$
$$N_2 = \frac{\xi(\xi+1)}{2}$$
$$N_3 = 1-\xi^2$$

From the slides of lecture 6, the equation to calculate the jacobian for a 1D element is the following:

$$\frac{\partial x}{\partial \xi} = \sum_{i=1}^{3} x_i \frac{\partial N_i}{\partial \xi}$$

From equation(7.1), and replacing x_i and N_i by their corresponding values, we get:

$$x = \frac{\xi(\xi+1)}{2} l + \left(\frac{l}{2} + \alpha l\right)(1-\xi^2)$$

Therefore, the jacobian is the following:

$$J = \frac{2\xi+1}{2} l - \xi l - 2\xi\alpha l$$

$$J = \frac{l - 4\xi\alpha}{2}$$

Placing this equation bigger or equal to zero ($J \geq 0$), two cases are taken: $\xi = 1$ and $\xi = -1$. l is always positive, therefore it is removed from the equation. Putting both cases together, the following is obtained:

$$\frac{-1}{4} \leq \alpha \leq \frac{1}{4}$$

Also placing $\alpha = 0$ in the other equation, we get a constant value of the jacobian $J = \frac{l}{2}$.

2)

$$\boldsymbol{B} = J^{-1}\boldsymbol{N}$$

The jacobian is a scalar therfore the inverse of jacobian will be the following:

$$J^{-1} = \frac{2}{l - 4\xi\alpha}$$

Therefore the displacement matrix will be the following:

$$\boldsymbol{B} = J^{-1}\frac{d\boldsymbol{N}}{d\xi} = \frac{2}{l - 4\xi\alpha}\begin{bmatrix}\frac{2\xi-1}{2} & \frac{2\xi+1}{2} & -2\xi\end{bmatrix}$$

Assignment 4.2

1) Referring to chapter 5 in "The Finite Element Method Fifth Edition Volume 1: The Basis":

$$\boldsymbol{B} = \begin{bmatrix}B_i & B_j & B_k\end{bmatrix}$$

$$B_i = \begin{bmatrix} b_i & 0 \\ 0 & c_i \\ \frac{a_i}{r} + b_i + \frac{c_i z}{r} & 0 \\ c_i & b_i \end{bmatrix}$$

While the constants a_i, b_i and c_i are calculated as following:

$$\begin{aligned} a_i &= r_j z_k - r_k z_j \\ b_i &= z_j - z_k \\ c_i &= r_k - r_j \end{aligned}$$

Using the equations above:

$$\boldsymbol{B} = \begin{bmatrix} -b & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & -a & 0 & a \\ \frac{(a-r)b}{r} & 0 & \frac{br-az}{r} & 0 & \frac{az}{r} & 0 \\ 0 & -b & -a & b & a & 0 \end{bmatrix}$$

Therefore, the global stiffness matrix will be the following:

$$\boldsymbol{K} = 2\pi\bar{r}A\boldsymbol{B}^{\boldsymbol{T}}\boldsymbol{E}\boldsymbol{B}$$

While the values of A (area of the triangle) and $\bar{r}$ (r of centroid) are the following:

$$A = \frac{r_i(z_j - z_k) + r_j(z_k - z_i) + r_k(z_i - z_j)}{2} = \frac{ab}{2}$$

$$\bar{r} = \frac{r_i + r_j + r_k}{3} = \frac{2a}{3}$$

Therefore, the final value of the stiffness matrix is the following:

$$\boldsymbol{K} = \frac{2\pi a^2 bE}{3}\begin{bmatrix} b^2 + \left(\frac{(a-r)b}{r}\right)^2 & 0 & -b^2 + \frac{(a-r)(br-az)b}{r^2} & 0 & \frac{az(a-r)b}{r^2} & 0 \\ & \frac{b^2}{2} & \frac{ab}{2} & -\frac{b^2}{2} & -\frac{ab}{2} & 0 \\ & & \frac{a^2}{2} + b^2 + \left(\frac{br-az}{r}\right)^2 & -\frac{ab}{2} & \frac{az(br-az)}{r^2} - \frac{a^2}{2} & 0 \\ & & & a^2 + \frac{b^2}{2} & \frac{ab}{2} & -a^2 \\ & symm & & & \left(\frac{az}{r}\right)^2 + \frac{a^2}{2} & 0 \\ & & & & & a^2 \end{bmatrix}$$

2) Summing the rows 2, 4, and 6:

$$\begin{aligned} Row2 =& \frac{b^2}{2} + \frac{ab}{2} - \frac{ab}{2} - \frac{b^2}{2} = 0 \\ Row4 =& -\frac{b^2}{2} - \frac{ab}{2} + a^2 + \frac{b^2}{2} + \frac{ab}{2} - a^2 = 0 \\ Row6 =& -a^2 + a^2 = 0 \end{aligned}$$

summing rows 1,3 and 5:

$$\begin{aligned} Row1 =& b^2 + \left(\frac{(a-r)b}{r}\right)^2 - b^2 + \frac{(a-r)(br-az)b}{r^2} + \frac{az(a-r)b}{r^2} = \frac{(a-r)b}{r^2}(f_1(r,z)) \neq 0 \\ Row3 =& \frac{br-az}{r^2} f_2(r,z) \neq 0 \\ Row5 =& \frac{az}{r^2} f_3(r,z) \neq 0 \end{aligned}$$

The summations in rows 2,4 and 6 are equal to zero because they represent the terms for the deformation of the nodes of the element in the z direction, where there is no symmetry conditions.

On the other hand, the summations in rows 1,3 and 5 are not equal to zero because they represent the terms for the deformation of the nodes of the element in the radial direction, where there is the axisymmetry condition.

3) Following the same book, and using values from 1), the consistent force vector is calculated the following way:

$$\boldsymbol{f^e} = -2\pi \boldsymbol{b} \frac{\bar{r}A}{3}$$

$$\boldsymbol{f^e} = \frac{2a^2 b\pi}{3}\begin{bmatrix} 0 \\ g \end{bmatrix}$$

Reference:

O.C.Zienkiewicz,R.L.Taylor - The Finite Element Method Fifth Edition Volume 1: The Basis