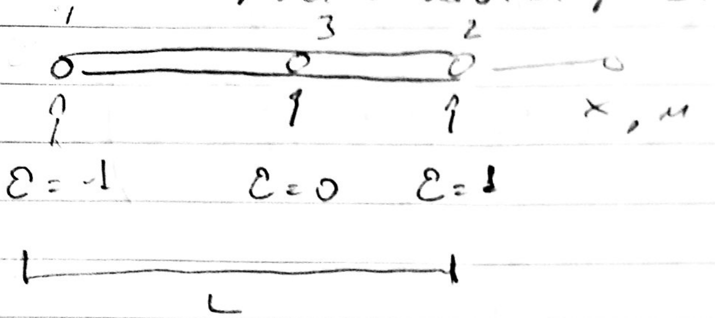
Assignment 4.1

AXIAL RIGIDITY EA



$x_1 = 0$

$x_2 = l$

$x_3 = l/2 + \alpha . l = l\left(\frac{1}{2} + \alpha\right) \quad \left(-\frac{1}{2} \leq \alpha \leq \frac{1}{2}\right)$

1. $N_1^l = C.(\varepsilon - 1).\varepsilon$

$N_1(-1) = 1$

$N_1(-1) = C . -2 . -1 = 2C$

$N_1(-1) = 1 = 2C$

$C = \frac{1}{2}$

$\rightarrow \underline{N_1^l(\varepsilon) = \frac{1}{2}(\varepsilon - 1).\varepsilon}$

$N_3^l = C(\varepsilon + 1)(\varepsilon - 1) \rightarrow \underline{N_3^l = -\varepsilon^2 + 1}$

$N_3(0) = 1 \rightarrow C = -1$

$N_2^l = C(\varepsilon + 1).\varepsilon$

$N_2(1) = 1 \rightarrow C = \frac{1}{2}$

$\rightarrow \underline{N_2 = \frac{1}{2}(\varepsilon + 1).\varepsilon}$

$x = N_1^l x_1 + N_2 x_2 + N_3 x_3$

$x = \frac{1}{2}(\varepsilon - 1).\varepsilon x_1 + (1 - \varepsilon^2).x_3 + \frac{1}{2}(\varepsilon + 1)\varepsilon\, x_2$

$(\mathcal{E}+1)\ 2(2\mathcal{E}+1)$

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$x_1 = 0 \qquad x_2 = l \qquad x_3 = \frac{l}{2} + \alpha l$

$x = \frac{l}{2}(\mathcal{E}+1).\mathcal{E} + (1-\mathcal{E}^2).\left(\frac{1}{2}+\alpha\right)l$

$x = l\left(\frac{\mathcal{E}^2+\mathcal{E}}{2} + \frac{1}{2} + \alpha - \frac{\mathcal{E}^2}{2} - \mathcal{E}^2\alpha\right)$

$x = l\left(\frac{\mathcal{E}}{2} + \frac{1}{2} + \alpha - \mathcal{E}^2\alpha\right)$

$J = \frac{dx}{d\mathcal{E}} = l\left(\frac{1}{2} + 0 + 0 - 2\mathcal{E}\alpha\right)$

$J = \frac{dx}{d\mathcal{E}} = \frac{l}{2}\left(1 - 4\mathcal{E}\alpha\right)$

$1 - 4\mathcal{E}\alpha > 0 \rightarrow J > 0$

$4\mathcal{E}\alpha < 1$

$\mathcal{E} = 1 \rightarrow \alpha < \frac{1}{4}$

$\mathcal{E} = -1 \rightarrow \alpha > -\frac{1}{4}$

$-1 < \mathcal{E} < 1 \rightarrow -\frac{1}{4} < \alpha < \frac{1}{4}$

if $\alpha = 0 \rightarrow J = \frac{l}{2}(1 - 4\mathcal{E}.0)$

$= \frac{l}{2}$

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2. $\varepsilon = \frac{du}{dx} = B.\bar{u}^e$

$$u_e = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \qquad u = \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$\frac{du}{dx} = \begin{bmatrix} \frac{dN_1}{dx}, & \frac{dN_2}{dx}, & \frac{dN_3}{dx} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$B = \frac{dN_1}{dx}, \frac{dN_2}{dx}, \frac{dN_3}{dx}$$

$$\frac{dN_1}{dx} = \frac{dN_1}{d\xi} \cdot \frac{d\xi}{dx} = \frac{dN_1}{d\xi} \cdot \left(\frac{dx}{d\xi}\right)^{-1}$$

$$\frac{dN_2}{dx} = \frac{dN_2}{d\xi} \cdot \frac{d\xi}{dx} = \frac{dN_2}{d\xi} \cdot \left(\frac{dx}{d\xi}\right)^{-1}$$

Same to dN_3/dx

$$B = J^{-1} \frac{dN}{d\xi}$$

$$B = \left(\frac{dx}{d\xi}\right)^{-1} \cdot \begin{bmatrix} \frac{dN_1}{d\xi}, & \frac{dN_2}{d\xi}, & \frac{dN_3}{d\xi} \end{bmatrix}$$

$$\frac{dN_1}{d\xi} = \left[\frac{1}{2}(\xi)(\xi-1)\right]' = (\xi - 1 + \xi)\frac{1}{2} = \xi - \frac{1}{2}$$

$$\frac{dN_2}{d\xi} = \left(\frac{1}{2}(\xi+1)\xi\right)' = \xi + \frac{1}{2}$$

$$\frac{dN_3}{d\xi} = \left(-\xi^2+1\right)' = -2\xi$$

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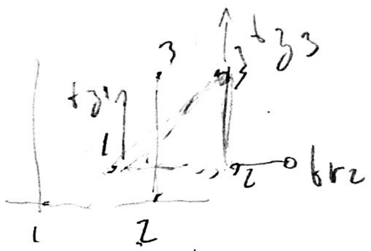

t_{r2} t_{z1} t_{z3}

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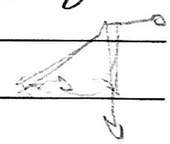
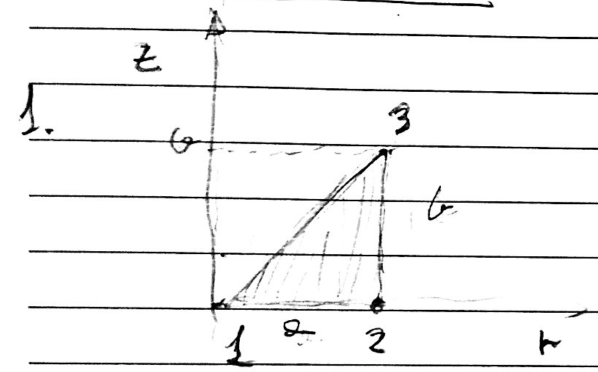
t_{r1} t_{r3} t_{z2}



$$B = \frac{2}{l(1-4\varepsilon\alpha)} \cdot \left[\varepsilon - \frac{1}{2},\ \varepsilon + \frac{1}{2},\ -2\varepsilon\right]$$

Assignment 4.2



$r_1 = 0 \qquad z_1 = 0$

$r_2 = l \qquad z_2 = 0$

$r_3 = l \qquad z_3 = b$

$$\begin{pmatrix} 1 \\ r \\ z \\ u_r \\ u_z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ r_1 & r_2 & r_3 \\ z_1 & z_2 & z_3 \\ u_{r1} & u_{r2} & u_{r3} \\ u_{z1} & u_{z2} & u_{z3} \end{pmatrix} \cdot \begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ r \\ z \\ u_r \\ u_z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & l & l \\ 0 & 0 & b \\ u_{r1} & u_{r2} & u_{r3} \\ u_{z1} & u_{z2} & u_{z3} \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{pmatrix}$$

$\varepsilon_2 + \varepsilon_3 = \frac{r}{l}$

$\varepsilon_3 = z$

$e = DNu^e \qquad e = Bu^e$ where $B = DN$

$\sigma = E\varepsilon$

$\mu_{r2} \quad \mu_{z1} \quad \mu_{z3} \rightarrow t_{r2} \quad t_{z1} \quad t_{z3}$

$\mu^e = (\mu_{r1} \; \mu_{r2} \; \mu_{r3} \; \mu_{z1} \; \mu_{z2} \; \mu_{z3})$

$$D = \begin{pmatrix} \frac{\partial}{\partial r} & 0 \\ 0 & \frac{\partial}{\partial z} \\ \frac{1}{r} & 0 \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial r} \end{pmatrix} \qquad N = \begin{pmatrix} \xi_1 & \xi_2 & \xi_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \xi_1 & \xi_2 & \xi_3 \end{pmatrix}$$

$(3 \times 6) \quad (6 \times$

$$l = \begin{pmatrix} l_{rr} \\ l_{zz} \\ l_{\theta\theta} \\ l_{rz} \end{pmatrix}$$

$B = D.N$

$$B = \begin{pmatrix} \frac{\partial \xi_1}{\partial r} & \frac{\partial \xi_2}{\partial r} & \frac{\partial \xi_3}{\partial r} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial \xi_1}{\partial z} & \frac{\partial \xi_2}{\partial z} & \frac{\partial \xi_3}{\partial z} \\ \frac{\xi_1}{r} & \frac{\xi_2}{r} & \frac{\xi_3}{r} & 0 & 0 & 0 \\ \frac{\partial \xi_1}{\partial z} & \frac{\partial \xi_2}{\partial z} & \frac{\partial \xi_3}{\partial z} & \frac{\partial \xi_1}{\partial r} & \frac{\partial \xi_2}{\partial r} & \frac{\partial \xi_3}{\partial r} \end{pmatrix}$$

3 1 2
2 3 1
1 2 3
i j k

$2A \cdot \frac{\partial \xi_i}{\partial r} = z_{jk} \qquad 2A \cdot \frac{\partial \xi_i}{\partial z} = r_{kj}$

$A = \underline{ab}$

$\frac{\partial \xi_1}{\partial r} = \frac{z_{23}}{2A} = \frac{z_2 - z_3}{2A} = \frac{z_2 - z_3}{ab} = \frac{-b}{ab} = \frac{-1}{a}$

$\frac{\partial \xi_1}{\partial z} = \frac{r_{32}}{2A} = \frac{r_{32}}{2A} = \frac{r_3 - r_2}{2A} = \frac{0}{2A} = 0$

$\frac{\partial \xi_2}{\partial r} = \frac{z_{31}}{2A} = \frac{z_3 - z_1}{2A} = \frac{b - 0}{ab} = \frac{1}{a}$

$\frac{\partial \xi_2}{\partial z} = \frac{r_{13}}{2A} = \frac{r_1 - r_3}{2A} = \frac{-a}{ab} = \frac{-1}{b}$

$\frac{\partial \xi_3}{\partial r} = \frac{z_{12}}{2A} = \frac{z_1 - z_2}{ab} = \frac{0}{ab} = 0$

$\frac{\partial \xi_3}{\partial z} = \frac{r_{21}}{2A} = \frac{r_2 - r_1}{ab} = \frac{a}{ab} = \frac{1}{b}$

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$\begin{matrix} u_r \\ u_\gamma \end{matrix} = N u^l$

$$B = \begin{bmatrix} -\frac{1}{a} & \frac{1}{a} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{b} & \frac{1}{b} \\ \frac{\xi_1}{r} & \frac{\xi_2}{r} & \frac{\xi_3}{r} & 0 & 0 & 0 \\ 0 & -\frac{1}{b} & \frac{1}{b} & -\frac{1}{a} & \frac{1}{a} & 0 \end{bmatrix}$$

$$B^T = \begin{bmatrix} -\frac{1}{a} & 0 & \frac{\xi_1}{r} & 0 \\ \frac{1}{a} & 0 & \frac{\xi_2}{r} & -\frac{1}{b} \\ 0 & 0 & \frac{\xi_3}{r} & \frac{1}{b} \\ 0 & 0 & 0 & -\frac{1}{a} \\ 0 & -\frac{1}{b} & 0 & \frac{1}{a} \\ 0 & \frac{1}{b} & 0 & 0 \end{bmatrix}$$

$$E = E \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$E\,B = E \begin{bmatrix} -\frac{1}{a} & \frac{1}{a} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{b} & \frac{1}{b} \\ \frac{\xi_1}{r} & \frac{\xi_2}{r} & \frac{\xi_3}{r} & 0 & 0 & 0 \\ 0 & -\frac{1}{2b} & \frac{1}{2b} & -\frac{1}{2a} & \frac{1}{2a} & 0 \end{bmatrix}$$

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$B^T \cdot E B$

$$
= \begin{bmatrix}
-\frac{1}{a^2} + \left(\frac{\xi_1}{r}\right)^2 & -\frac{1}{a^2} + \frac{\xi_1 \xi_2}{r^2} & \frac{\xi_1 \xi_3}{r^2} & 0 & 0 & 0 \\
-\frac{1}{a^2} + \frac{\xi_1 \xi_2}{r^2} & \frac{1}{a^2} + \frac{\xi_2^2}{r^2} + \frac{1}{2b^2} & \frac{\xi_2 \xi_3}{r^2} - \frac{1}{2b^2} & \frac{\pm 1}{2ab} & \frac{-1}{2ab} & 0 \\
\frac{\xi_3 \xi_1}{r^2} & \frac{\xi_2 \xi_3}{r^2} - \frac{1}{2b^2} & \frac{\xi_3^2}{r^2} + \frac{1}{2b^2} & \frac{-1}{2ab} & \frac{1}{2ab} & 0 \\
0 & +\frac{1}{2ab} & -\frac{1}{2ab} & \frac{1}{2a^2} & -\frac{1}{2a^2} & 0 \\
0 & -\frac{1}{2ab} & \frac{1}{2ab} & -\frac{1}{2a^2} & \frac{1}{2a^2} + \frac{1}{b^2} & -\frac{1}{b^2} \\
0 & 0 & 0 & 0 & -\frac{1}{b^2} & \frac{1}{b^2}
\end{bmatrix}
$$

$K_1 = \int_{\Omega^e} B^T E B \, r \, dA$ - Using numerical quadrature

Mid point rule: $\frac{1}{A} \int_{\Omega_e} F(\xi_1, \xi_2 \xi_3) \cong \frac{1}{3} F\left(\frac{1}{2}, \frac{1}{2}, 0\right) + \frac{1}{3} F\left(\frac{1}{2}, 0, \frac{1}{2}\right) + F\left(0, \frac{1}{2}, \frac{1}{2}\right)$

$r_1\left(\frac{1}{2}, \frac{1}{2}, 0\right) = \sum_{m}^{3} \xi_m r_m = 0 \cdot \frac{1}{2} + a \cdot \frac{1}{2} + a \cdot 0 = \frac{a}{2}$

$r_2\left(\frac{1}{2}, 0, \frac{1}{2}\right) = 0 \cdot \frac{1}{2} + a \cdot 0 + a \cdot \frac{1}{2} = \frac{a}{2}$

$r_3\left(0, \frac{1}{2}, \frac{1}{2}\right) = 0 \quad 0 + a \cdot \frac{1}{2} + a \cdot \frac{1}{2} = a$

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$F(\varepsilon_1, \varepsilon_2, \varepsilon_3, r) = B^T \varepsilon B (\varepsilon_1, \varepsilon_2, \varepsilon_3, r) \cdot r$

$F_1 = \frac{1}{3} \cdot F(\frac{1}{2}, \frac{1}{2}, 0, \frac{e}{2}) = \frac{1}{3} B^T \varepsilon B (\frac{1}{2}, \frac{1}{2}, 0, \frac{e}{2}) \cdot \frac{e}{2}$

$$F_1 = \frac{e}{6} \begin{bmatrix} \frac{2}{e^2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{e^2} + \frac{1}{2b^2} & -\frac{1}{2b^2} & \frac{1}{2eb} & -\frac{1}{2eb} & 0 \\ 0 & -\frac{1}{2b^2} & \frac{1}{2b^2} & -\frac{1}{2eb} & \frac{1}{2eb} & 0 \\ 0 & \frac{1}{2eb} & -\frac{1}{2eb} & \frac{1}{2e^2} & -\frac{1}{2e^2} & 0 \\ \vdots & & & & & \end{bmatrix}$$

$F_2 = \frac{1}{3} B^T \varepsilon B (\frac{1}{2}, 0, \frac{1}{2}, \frac{e}{2}) \cdot \frac{e}{2}$

$$F_2 = \frac{e}{6} \begin{bmatrix} \frac{2}{e^2} & -\frac{1}{e^2} & \frac{1}{e^2} & 0 & 0 & 0 \\ 0 & \frac{1}{e^2} + \frac{1}{2b^2} & -\frac{1}{b^2} & \frac{1}{2eb} & -\frac{1}{2eb} & 0 \\ \frac{1}{e^2} & -\frac{1}{2b^2} & \frac{1}{e^2} + \frac{1}{2b^2} & -\frac{1}{2eb} & \frac{1}{2eb} & 0 \end{bmatrix}$$

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$$F_3 = \frac{1}{3} B^T E B \left(0, \frac{1}{2}, \frac{1}{2}, e\right) \cdot e$$

$$F_3 = \frac{e}{3} \begin{bmatrix} \frac{1}{e^2} & -\frac{1}{e^2} & 0 & 0 & 0 & 0 \\ -\frac{1}{e^2} & \frac{2}{e^2} + \frac{1}{2b^2} & \frac{1}{e^2} - \frac{1}{2b^2} & \frac{1}{2eb} & -\frac{1}{2eb} & 0 \\ 0 & \frac{1}{e^2} - \frac{1}{2b^2} & \frac{1}{e^2} + \frac{1}{2b^2} & -\frac{1}{2eb} & \frac{1}{2eb} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

$$F = F_1 + F_2 + F_3$$

$$K = \frac{e \cdot b}{2} \left(F_1 + F_2 + F_3\right) = \frac{e^2 b}{12} F_1' + \frac{e^2 b}{12} F_2' + \frac{e^2 b}{6} F_3'$$

$$K = \begin{bmatrix} \frac{b}{2} & -\frac{b}{12} & \frac{b}{12} & 0 & 0 & 0 \\ -\frac{b}{6} & \frac{7b}{12} + \frac{e^2}{6b} & -\frac{5e^2}{24b} + \frac{b}{6} & \frac{e}{6} & -\frac{e}{6} & 0 \\ \frac{b}{12} & -\frac{e^2}{6b} + \frac{b}{6} & \frac{e^2}{6b} + \frac{b}{4} & -\frac{e}{6} & \frac{e}{6} & 0 \\ 0 & \frac{e}{6} & -\frac{e}{6} & \frac{b}{6} & -\frac{b}{6} & 0 \\ 0 & -\frac{e}{6} & \frac{e}{6} & -\frac{b}{6} & \frac{b}{6} + \frac{e^2}{36} & -\frac{e^2}{36} \\ 0 & 0 & 0 & 0 & -\frac{e^2}{36} & \frac{e^2}{36} \end{bmatrix}$$

3 consistent force

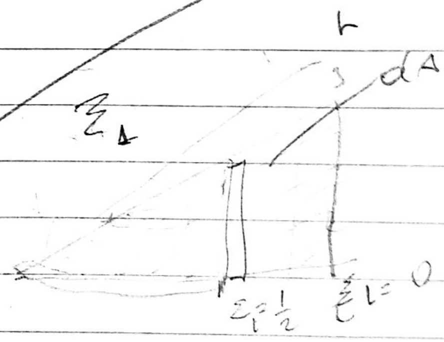
$$f_{ext} = \int N^T \cdot b \; r \, dA$$

$$b = [0, -g]^T$$

$$b = \begin{bmatrix} 0 \\ -g \end{bmatrix}$$

$$N^T \cdot b = \begin{bmatrix} \varepsilon_1 & 0 \\ \varepsilon_2 & 0 \\ \varepsilon_3 & 0 \\ 0 & \varepsilon_1 \\ 0 & \varepsilon_2 \\ 0 & \varepsilon_3 \end{bmatrix} \times \begin{bmatrix} 0 \\ -g \end{bmatrix} = -g \begin{bmatrix} 0 \\ 0 \\ 0 \\ \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix}$$

$$f_{ext} = \int_{\Omega^e} -g r \begin{bmatrix} 0 \\ 0 \\ 0 \\ \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} dA$$



$$dA = \frac{r b}{e} \cdot dr$$

$$\int_{\Omega} -g r \, \varepsilon_1 \, dA$$

$\boxed{\text{Analytically for } f_{y1}}$

$\varepsilon_1 = 1 \quad r = 0$

$\varepsilon_1 = 0 \quad r = e$

$$\int_0^e -g r \left(-\frac{r}{e} + 1\right) \cdot \frac{r b}{e} \, dr$$

$$= \int_0^e \left(+g \frac{r^3 b}{e^2} - g \frac{r^2 b}{e} \right) dr$$

$\varepsilon = A r + b \qquad \varepsilon = -\frac{r}{e} + 1$

$1 = 0 + b$

$0 = A \cdot e + 1$

$A = -\frac{1}{e}$

$$\frac{bg}{e} \int_0^e \left(\frac{r^3}{e} - r^2 \right) dr$$

$$= \frac{bg}{e} \left(\frac{e^4}{4e} - \frac{e^3}{3} \right) = bg \left(\frac{e^2}{4} - \frac{e^2}{3} \right) = -\frac{bg e^2}{12}$$

$$f = -g \int \begin{bmatrix} 0 \\ 0 \\ 0 \\ \varepsilon_1 r \\ \varepsilon_2 r \\ \varepsilon_3 r \end{bmatrix} dA$$

$\boxed{\text{Numericaly}}$

Aplying numerical quadrature - midpoint rule (3 points)

$$\approx \frac{1}{3}\left[F\left(\tfrac{1}{2}, \tfrac{1}{2}, \varepsilon, r_1\right) + F\left(\tfrac{1}{2}, 0, \tfrac{1}{2}, r_2\right) + F\left(0, \tfrac{1}{2}, \tfrac{1}{2}, r_3\right) \right]$$

$r_1 = \frac{a}{2}$ $\quad r_2 = \frac{a}{2}$ $\quad r_3 = a$

$$\approx \frac{1}{3}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{a}{4} \\ \frac{a}{4} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{a}{4} \\ 0 \\ \frac{a}{4} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{a}{2} \\ \frac{a}{2} \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ a/6 \\ a/4 \\ a/4 \end{bmatrix}$$

$$f \approx \begin{bmatrix} 0 \\ 0 \\ 0 \\ -g\frac{a^2 b}{12} \\ -g\frac{a^2 b}{8} \\ -g\frac{a^2 b}{8} \end{bmatrix}$$

— checks with analitical computations