Computational Structural Mechanics and Dynamics

Masters in Numerical Methods

Assignment 4

Isoparametric representation and Structures of Revolution

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1 Assignment 4.1

1.1 Part A

Here we have to compute the Jacobian of the element. We define

$$x_1 = 0 \qquad x_2 = l \qquad x_2 = (1/2 + \alpha)l \tag{1}$$

The shape functions for a three noded linear element are given by

$$N_1^e = \frac{\xi}{2}(1-\xi) \qquad N_2^e = \frac{\xi}{2}(1+\xi) \qquad N_2^e = 1-\xi^2 \tag{2}$$

The x coordinate of a point within the element is written in the iso-parametric formulation as

$$x = \sum N_i(\xi)x_i \tag{3}$$

Substituting the values of the shape functions and the values of the x_i :

$$x = \frac{\xi}{2}(1-\xi)(0) + \frac{\xi}{2}(1+\xi)l + 1 - \xi^2((\frac{1}{2}+\alpha)l \tag{4}$$

$$x = -\frac{l}{2}(2\xi^2\alpha - \xi - 2\alpha - 1) \tag{5}$$

The Jacobian is defined as :

$$J = \frac{\partial x}{\partial \xi} = \frac{l}{2}[1 - 4\xi\alpha] \tag{6}$$

It can be clearly seen that $J = \frac{l}{2}$ if $\alpha = 0$, this is the case where x_3 is the midpoint of the element and this result matches with the value of the Jacobian obtained in that case.

It can also be seen that, $J > 0$ if $4\xi\alpha < 1$ now if $-1 < \xi < 1$ we can conclude that $\alpha \in (-\frac{1}{4}, \frac{1}{4})$

1.2 Part B

The strain matrix **B** is defined as

$$\mathbf{B} = \frac{\partial \xi}{\partial x}[\frac{dN_1}{d\xi}, \frac{dN_2}{d\xi}, \frac{dN_3}{d\xi}] \tag{7}$$

In this case, it is given by

$$\mathbf{B} = \frac{2}{(1-4\xi\alpha)l^e}[\xi - \frac{1}{2}, -2\xi, \xi + \frac{1}{2}] \tag{8}$$

2 Assignment 4.2

2.1 Part A

In this section we have to find the stiffness matrix for an axisymmetric triangle whose coordinates are as follows: (r,z) = $[(0,0),(a,0),(a,b)]$
The material is isotropic with $\nu = 0$ and the Elasticity matrix is as follows :

$$\mathbf{E} = E \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix} \tag{9}$$

The shape functions are given by

$$N_1 = 1 - \frac{r}{a} \qquad N_2 = \frac{r}{a} - \frac{z}{b} \qquad N_3 = \frac{z}{b} \tag{10}$$

The stiffness matrix is given by the equation

$$K_{ij}^e = 2\pi \int\int_{A^e} B_i^T D B_j r dr dz \tag{11}$$

The **B** matrix is given by $\mathbf{B} = [\boldsymbol{B_1}, \boldsymbol{B_2}, \boldsymbol{B_3}]$, where $\boldsymbol{B_i}$ is given by:

$$\boldsymbol{B_i} = \begin{bmatrix} \frac{dN_i}{dr} & 0 \\ 0 & \frac{dN_i}{dz} \\ \frac{N_i}{r} & 0 \\ \frac{dN_i}{dz} & \frac{dN_i}{dr} \end{bmatrix} \tag{12}$$

Instead of integrating a much simpler approximation is to compute the integral at the centroid of the triangle the centroid is given by $(\bar{r}, \bar{z}) = (\frac{2a}{3}, \frac{b}{3})$. The stiffness matrix equation then becomes :

$$K_{ij}^e = 2\pi \bar{B_i^T} D \bar{B}_j \bar{r} \Delta \tag{13}$$

here $\Delta = \frac{ab}{2}$ is the area of the triangle.
The stiffness matrix is given by

$$K = \frac{\pi E}{6b} \begin{bmatrix} 5b^2 & 0 & -3b^2 & 0 & b^2 & 0 \\ 0 & 2b^2 & 2ab & -2b^2 & -2ab & 0 \\ -3b^2 & 2ab & 2a^2+5b^2 & -2ab & b^2-2a^2 & 0 \\ 0 & -2b^2 & -2ab & 2(b^2+2a^2) & 2ab & -4a^2 \\ b^2 & -2ab & b^2-2a^2 & 2ab & b^2+2a^2 & 0 \\ 0 & 0 & 0 & -4a^2 & 0 & 4a^2 \end{bmatrix} \tag{14}$$

2.2 Part B

The sum of rows and columns 2,4 and 6 vanish because they represent the balance of forces in the z direction. The forces in the z direction must be balanced. The forces in the radial direction are not balanced because this direction is perpendicular to the direction of symmetry. Thus any movement in this direction will generate asymmetric stresses. This asymmetric may vanish in a 3D model of this problem since we will consider complete symmetry.

2.3 Part C

The force vector is given by

$$f^e = \int N^T b r dA \tag{15}$$

The body force vector is given by b = $\begin{bmatrix} 0 \\ -g \end{bmatrix}$

The **N** matrix is given by

$$\boldsymbol{N^T} = \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \end{bmatrix} \tag{16}$$

Just as we did for the stiffness matrix the forcing vector is calculated at the centroid of the triangle.

$$f^e = -2\pi \bar{N}^T b \bar{r} \Delta \tag{17}$$

The force vector is given by

$$f^e = \frac{-2\pi a^2 b g}{9} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \tag{18}$$

3 Appendix

A MATLAB code for symbolic matrix multiplication written for this assignment.

```
close all
clear all

a = sym('a');
b = sym('b');
E = sym('E');
g = sym('g');

x = [0,a,a];
y = [0,0,b];

Area = a*b/2;

rbar = (2*a)/3;

B = [-1/a,0,1/a,0,0,0;0,0,0,-1/b,0,1/b;1/(2*a),0,1/(2*a),0,1/(2*a
    ),0;0,-1/a,-1/b,1/a,1/b,0];

D = [1,  0,  0,0;  0,1,0,0;0,0,1,0;  0,0,0,1/2];

k1 = mtimes(D,B);
K = mtimes(transpose(B),k1);


Knew = rbar*Area*K;
disp(Knew)
```