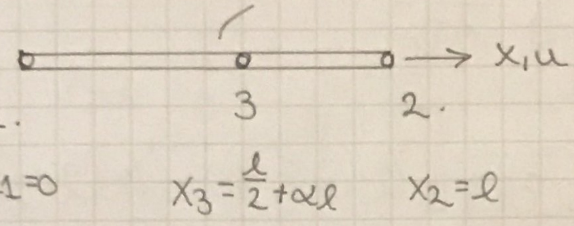
Assignment 4 - Computational Structural Mechanics and dynamics

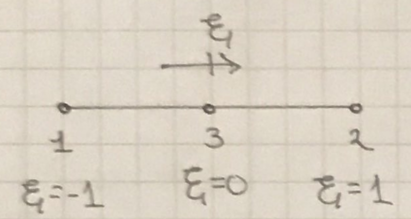
Assignment 4,1

3-node straight bar



1)

Uses reference-element to obtain the shape functions



usen Langrange polynomials

$$\Rightarrow N_1 = \frac{(\xi_1 - 0)(\xi_1 - 1)}{(-1)(-1-1)} = \underline{\tfrac{1}{2}\xi_1(\xi_1 - 1)}$$

$$N_3 = \frac{(\xi_1 + 1)(\xi_1 - 1)}{(1)(-1)} = \underline{1 - \xi_1^2} \qquad N_2 = \frac{(\xi_1 + 1)(\xi_1 + 0)}{(1+1)(1+0)} = \underline{\tfrac{1}{2}\xi_1(\xi_1 + 1)}$$

$$x = \sum N_i x_i = N_1 x_1 + N_2 x_2 + N_3 x_3$$

$$= \frac{\xi_1 \cdot l}{2}(\xi_1 + 1) + (1 - \xi_1^2)\left[\, l/2 + \alpha \cdot l \,\right]$$

$$\Rightarrow \text{Jacobian} = \frac{\partial x}{\partial \xi_1} = \frac{l}{2}(2\xi_1 + 1) - 2\xi_1\left[\, l/2 + \alpha \cdot l \,\right]$$

If $-1/4 < \alpha < 1/4$ then $J > 0$ for whole element

$\underline{\alpha = -1/4:}$

$$J = l/2\,(2\xi_1 + 1) - 2\xi_1 \underbrace{\left(l/2 - \frac{l}{4} \right)}_{l/2} = \frac{l}{2}\left[\cancel{2\xi_1} + 1 - \cancel{2\xi_1}\right] = \underline{\frac{l}{2}}$$

$\underline{J = l(1/2 - 2\xi_1\alpha) \Rightarrow \text{will always be} > 0 \text{ if } 1/4 < \alpha < \frac{1}{4}}$

$\alpha = 0: \Rightarrow \underline{J = \ell/2} \Rightarrow$ constant over the whole element.

2) Strain displacement matrix $\mathbb{B}$ relating $\varepsilon = \frac{du}{dx}$

$$\mathbb{B} = \frac{dN}{dx} = \frac{1}{J}\frac{dN}{d\xi_1}$$

$$\Rightarrow \frac{1}{\ell(1/2 - 2\xi_1\alpha)}\left[\underbrace{1/2(2\xi_1 - 1)}_{\frac{\partial N_1}{\partial \xi_1}} \quad \underbrace{1/2(2\xi_1 + 1)}_{\frac{\partial N_2}{\partial \xi_1}} \quad \underbrace{-2\xi_1}_{\frac{\partial N_3}{\partial \xi_1}}\right]$$

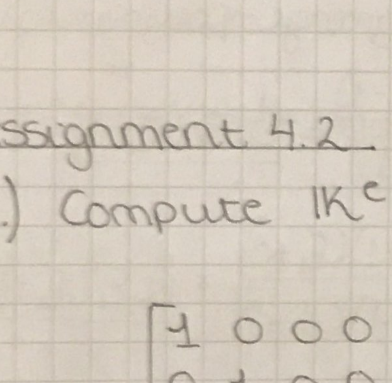
Assignment 4.2

1.) Compute $\mathbb{K}^e \Rightarrow r_1 = 0,\ r_2 = r_3 = a$

$z_1 = z_2 = 0,\ z_3 = b$

$$\mathbb{E} = E\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$$

Axisymmetric triangle:



$$\begin{bmatrix} 1 \\ r \\ z \\ u_r \\ u_z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & a & a \\ 0 & 0 & b \\ u_{r1} & u_{r2} & u_{r3} \\ u_{z1} & u_{z2} & u_{z3} \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix}$$

assumed in Area-coordinates

$$\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \frac{1}{2A}\begin{bmatrix} r_2 z_3 - z_2 r_3 & z_{23} & r_{32} \\ r_3 z_1 - z_3 r_1 & z_{31} & r_{13} \\ r_1 z_2 - z_1 r_2 & z_{12} & r_{21} \end{bmatrix}\begin{bmatrix} 1 \\ r \\ z \end{bmatrix}$$

where

$r_{ij} = r_i - r_j$

$z_{ij} = z_i - z_j$

This is relation between Area coordinates and axi-symmetric coordinates.

$$\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \frac{1}{ab} \begin{bmatrix} ab & -b & 0 \\ 0 & b & -a \\ 0 & 0 & a \end{bmatrix} \begin{bmatrix} 1 \\ r \\ z \end{bmatrix}$$

area coords

$N_1 = \xi_1$

$$\underline{\xi_1 = \frac{1}{ab}[ab - br] = \frac{1}{a}[a-r] = N_1}$$

$$\underline{\xi_2 = \frac{1}{ab}[br - az] = N_2}$$

$$N = \begin{bmatrix} N_1 & N_2 & N_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & N_1 & N_2 & N_3 \end{bmatrix}$$

$$\underline{\xi_3 = \frac{1}{ab}[az] = \frac{z}{b} = N_3}$$

$$\mathbb{D} = \begin{bmatrix} \partial/\partial r & 0 \\ 0 & \partial/\partial z \\ 1/r & 0 \\ \partial/\partial z & \partial/\partial r \end{bmatrix} \Rightarrow \mathbb{B} = \mathbb{D}N = \begin{bmatrix} \Phi_r & 0 \\ 0 & \Phi_z \\ \Phi_\theta & 0 \\ \Phi_z & \Phi_r \end{bmatrix}$$

$$\Phi_r = \left[\frac{\partial N_1}{\partial r} \quad \frac{\partial N_2}{\partial r} \quad \frac{\partial N_3}{\partial r}\right] \qquad \Phi_\theta = \left[\frac{N_1}{r} \quad \frac{N_2}{r} \quad \frac{N_3}{r}\right]$$

$$\Phi_z = \left[\frac{\partial N_1}{\partial z} \quad \frac{\partial N_2}{\partial z} \quad \frac{\partial N_3}{\partial r}\right]$$

$$\mathbb{B} = \begin{bmatrix} -1/a & 1/a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1/b & 1/b \\ \frac{1}{r} - 1/a & \frac{1}{a} - \frac{z}{br} & \frac{z}{br} & 0 & 0 & 0 \\ 0 & -1/b & 1/b & -1/a & 1/a & 0 \end{bmatrix}$$

$$\mathbb{K}^e = \int_V \mathbb{B}^T \mathbb{E} \mathbb{B}\, dV = \int \mathbb{B}^T \mathbb{E} \mathbb{B}\, r\, dr\, dz\, d\theta$$

$$= 2\pi \int_0^a \int_0^{\frac{b}{a} r} \mathbb{B}^T \mathbb{E} \mathbb{B}\, r\, dr\, dz \Rightarrow \begin{array}{l} 0 \le r \le a \\ 0 \le z \le \frac{b}{a} r \end{array}$$

can solve this analytically or by Numerical integration by Gauss rules

$\Rightarrow$ choose analytically by the help of MathLAB:

$$\mathbb{K}^e = 2\pi E \begin{bmatrix} 2b/3 & -b/4 & b/12 & 0 & 0 & 0 \\ -b/4 & a^2/6b + 4b/9 & -\frac{a^2}{6b} + b/18 & a/6 & -a/6 & 0 \\ b/12 & -a^2/6b + \frac{b}{18} & a^2/6b + b/9 & -a/6 & a/6 & 0 \\ 0 & a/6 & -a/6 & b/6 & -b/6 & 0 \\ 0 & -a/6 & a/6 & -b/6 & \frac{a^2}{3b} + b/6 & -\frac{a^2}{3b} \\ 0 & 0 & 0 & 0 & -\frac{a^2}{3b} & \frac{a^2}{3b} \end{bmatrix}$$

2) The orientation of the rows and columns in my $\mathbb{K}^e$ is a little different due to

$$\mathbb{N} = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & N_1 & N_2 & N_3 & N_4 \end{bmatrix}$$

– same as the one in the PP-presentation.

Therefore will column/row 1-3-5 be raw/col 4-5-6 in this $\mathbb{K}^e$.

column:

$$\begin{bmatrix} 0 \\ a/6 - a/6 \\ -a/6 + a/6 \\ b/6 - b/6 \\ -b/6 + \frac{a^2}{3b} + b/6 - \frac{a^3}{3b} \\ -a^2/3b + a^2/3b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

ok!

Same for the rows due to a symmetric $\mathbb{K}^e$.

Row/col 4-5-6 are linearly dependent because
$R_4 + R_5 = R_6$.

Row/col 4-5-6 corresponds to the translation of the element in z-direction. Due to lack of constraints in z-direction, it is allowed to translation ⇒ sum equals zero.

$$\text{For this } \mathbb{K}^e \Rightarrow \overset{\text{row/col}}{(1,3;5)} = \overset{\text{row/col}}{(1,2,3)}$$

Sum column

$$\begin{bmatrix} \frac{2b}{3} - b/4 + b/12 \\ b/4 + \cancel{\frac{a^2}{6b}} + \frac{4b}{9} - \cancel{\frac{a^2}{6b}} + b/18 \\ b/12 - \cancel{\frac{a^2}{6b}} + b/18 + \cancel{\frac{a^2}{6b}} + b/9 \\ a/6 - a/6 \\ -a/6 + a/6 \\ 0 \end{bmatrix} = \begin{bmatrix} b/2 \\ b/4 \\ b/4 \\ 0 \\ 0 \\ 0 \end{bmatrix} \neq 0 \quad \text{not zero}$$

Column responds to translation of the element in the r-direction. The axisymmetric element is a "ring element", so a translation in r-direction would result in an anti-symmetric "ring". Therefore there are constraints in r-direction ⇒ sum ≠ 0.

3) Compute the consistent force vector $\mathbb{f}_e$ for gravity forces $\mathbb{b} = [0, -g]^T$

$$\mathbb{f}_e = \int_A \mathbb{N}^T \mathbb{b} \, r \, dA = 2\pi \int_A \frac{1}{2A} \begin{bmatrix} ab-br & 0 \\ br-az & 0 \\ az & 0 \\ 0 & ab-br \\ 0 & br-az \\ 0 & az \end{bmatrix} \begin{bmatrix} 0 \\ -g \end{bmatrix} r \, dA$$

Solve it nummerical

$$\Rightarrow \mathbb{f}_e \simeq \frac{1}{2} \sum_{k=1}^{p} \sum_{l=1}^{q} w_k w_l \, r_{k,l} \, \mathbb{N}_{k,l}^T \, \mathbb{b} \, |\mathbb{J}_{k,l}|$$

$$\mathbb{J} = \begin{bmatrix} \frac{\partial r}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial r}{\partial \eta} & \frac{\partial z}{\partial \eta} \end{bmatrix} = \begin{bmatrix} -a & -b \\ 0 & -b \end{bmatrix}$$

Uses one gauss point $\Rightarrow j=1, k=1$

$|\mathbb{J}_{k,l}| = ab$

$$\Rightarrow r = \frac{2a}{3}, \quad z = \frac{b}{3}$$

$$ab \cdot \frac{1}{2} \cdot \frac{2a}{3} \begin{bmatrix} 1/3\,ab & 0 \\ 1/3\,ab & 0 \\ 1/3\,ab & 0 \\ 0 & 1/3\,ab \\ 0 & 1/3\,ab \\ 0 & 1/3\,ab \end{bmatrix} \begin{bmatrix} 0 \\ -g \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{a^2bg}{9} \\ -\frac{a^2bg}{9} \\ -\frac{a^2bg}{9} \end{bmatrix} \frac{2\pi}{ab} \cdot ab$$

$$= \frac{2\pi a^2 bg}{9} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ -1 \\ -1 \end{bmatrix}$$

2π vanish with the 2π from $\mathbb{K}^e$.