Computational Structural Mechanics and Dynamics (Assignment 4)

Prakhar Rastogi (Masters in Computational Mechanics)

9th March 2020

On "Isoparametric representation"

Assignment 4.1

A 3-node straight bar element is defined by 3 nodes: 1, 2 and 3 with axial coordinates x_1, x_2 and x_3 respectively as illustrated in figure below. The element has axial rigidity EA, and length $l = x_1 - x_2$. The axial displacement is u(x). The 3 degrees of freedom are the axial node displacement u_1, u_2 and u_3. The isoparametric definition of the element is

$$\begin{bmatrix} 1 \\ x \\ u \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ N_3^e \end{bmatrix} \tag{7.1}$$

in which $N^e_1(\xi)$ are the shape functions of a three bar element. Node 3 lies between 1 and 2 but is not necessarily at the midpoint $x = l/2$. For convenience define,

$$x_1 = 0 \qquad x_2 = l \qquad x_3 = (\tfrac{1}{2} + \alpha)l \tag{7.2}$$

where $-½ < \alpha < ½$ characterizes the location of node 3 with respect to the element center. If $\alpha=0$ node 3 is located at the midpoint between 1 and 2.

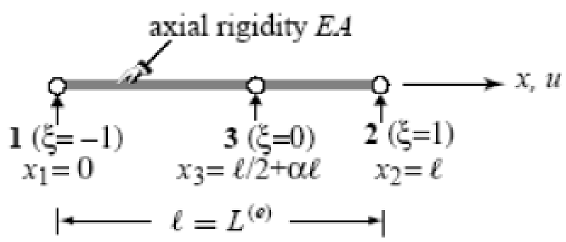

Figure.- The three-node bar element in its local system

1. From (7.2) and the second equation of (7.1) get the Jacobian $J = dx/d\xi$ in terms of l, α and ξ. Show that,

- if $-¼ < \alpha < ¼$ then $J > 0$ over the whole element $-1 \leq \xi \leq 1$.
- if $\alpha = 0$, $J = l/2$ is a constant over the element.

2. Obtain the 1x3 strain displacement matrix **B** relating $e = du/dx = \mathbf{B}\mathbf{u}^e$ where $\mathbf{u}^e$ is the column 3-vector of the node displacement u_1, u_2 and u_3. The entries of **B** are functions of l, α and ξ.

Hint: $\mathbf{B} = d\mathbf{N}/dx = J^{-1}d\mathbf{N}/d\xi$, where $\mathbf{N} = [\, N_1\ N_2\ N_3\,]$ and J comes from item a).

Solution :

1.

$$J = \frac{dx}{d\xi} = \frac{dN_1}{d\xi}x_1 + \frac{dN_2}{d\xi}x_2 + \frac{dN_3}{d\xi}x_3$$

$N_1 = \frac{1}{2}\xi(\xi - 1)$ $\quad N_2 = \frac{1}{2}\xi(\xi + 1)$ $\quad N_3 = (1 - \xi^2)$

$\frac{dN_1}{d\xi} = \frac{2\xi - 1}{2}$ $\quad \frac{dN_2}{d\xi} = \frac{2\xi + 1}{2}$ $\quad \frac{dN_3}{d\xi} = (-2\xi)$

$x_1 = 0$ $\quad x_2 = l$ $\quad x_3 = \frac{l}{2} + \alpha l$

$$\frac{dx}{d\xi} = \frac{dN_1}{d\xi}x_1 + \frac{dN_2}{d\xi}x_2 + \frac{dN_3}{d\xi}x_3 = \frac{1}{2}(2\xi + 1)(l) - 2\xi(\frac{l}{2} + \alpha l) = \frac{l}{2}[1 - 4\xi\alpha]$$

- If J>0, then $\frac{l}{2}[1 - 4\xi\alpha]>0$ $\quad \xi\alpha < 1/4$

Consider, -1/4<$\xi < 1/4$. If $\alpha = -1/4, \quad \xi > -1 \quad and \quad \alpha = 1/4 \quad signifies \quad \xi < 1$

Therefore,-1<$\xi < 1$ for J>0 and -1/4<$\xi < 1/4$

- $For\ \alpha = 0,\ J = \frac{l}{2}(1 - 4\xi\alpha) = \frac{l}{2}(1 - 4\xi(0)) = \frac{l}{2}$

2. dN/dξ = $\begin{pmatrix} \frac{dN_1}{d\xi} & \frac{dN_2}{d\xi} & \frac{dN_3}{d\xi} \end{pmatrix}$

$$B = \frac{dN}{dx} = J^{-1}\frac{dN}{d\xi} = \frac{2}{l(1 - 4\xi\alpha)}\begin{pmatrix} \frac{1}{2}(2\xi - 1) & \frac{1}{2}(2\xi + 1) & 1 - \xi^2 \end{pmatrix}$$

On "Structures of revolution"

Assignment 4.2

1. Compute the entries of $\mathbf{K}^e$ for the following axisymmetric triangle:

$r_1 = 0 \qquad r_2 = r_3 = a, \quad z_1 = z_2 = 0, \quad z_3 = b$

The material is isotropic with ν = 0 for which the stress-strain matrix is,

$$\boldsymbol{E} = E\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

2. Show that the sum of the rows (and columns) 2, 4 and 6 of $\mathbf{K}^e$ must vanish and explain why. Show as well that the sum of rows (and columns) 1, 3 and 5 does not vanish, and explain why.

3. Compute the consistent force vector $\mathbf{f}^e$ for gravity forces $\mathbf{b} = [0, -g]^T$.

1. As first approximation,

$$[K] = 2\pi \int B^{\mathrm{T}} E B r dr dz$$

$$[K] = 2\pi\tilde{r}A[\tilde{B}]^{\mathrm{T}}[\tilde{E}][\tilde{B}]$$

Centroidal coordinates (,) are
$\tilde{r} = (r_1 + r_2 + r_3)/3 = 2a/3$ $\tilde{z} = (z_1 + z_2 + z_3)/3 = b/3$

$$\epsilon = Ba^e = [B_1, B_2, B_3]a^e$$

$$[B_1] = \begin{pmatrix} \partial N_1/\partial r & 0 \\ 0 & \partial N_1/\partial z \\ N_1/\tilde{r} & 0 \\ \partial N_1/\partial z & \partial N_1/\partial r \end{pmatrix} = \begin{pmatrix} \beta_1 & 0 \\ 0 & \gamma_1 \\ \frac{\alpha_1}{\tilde{r}} + \beta_1 + \frac{\gamma_1 \tilde{z}}{\tilde{r}} & 0 \\ \gamma_1 & \beta_1 \end{pmatrix}$$

Similarly, we can calculate [B$_2$]$and[B_3]$.
$\alpha_1 = r_2 z_3 - z_3 r_2 = ab$ $\alpha_2 = r_3 z_1 - z_1 r_3 = 0$ $\alpha_3 = r_1 z_2 - z_2 r_1 = 0$
$\beta_1 = z_2 - z_3 = -b$ $\beta_2 = z_3 - z_1 = b$ $\beta_3 = z_1 - z_2 = 0$
$\gamma_1 = r_2 - r_3 = 0$ $\gamma_2 = r_3 - r_1 = -a$ $\gamma_3 = r_1 - r_2 = a$

$$[\tilde{B}] = \frac{1}{2|A|} \begin{pmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \frac{\alpha_1}{\tilde{r}} + \beta_1 + \frac{\gamma_1 \tilde{z}}{\tilde{r}} & 0 & \frac{\alpha_2}{\tilde{r}} + \beta_2 + \frac{\gamma_2 \tilde{z}}{\tilde{r}} & 0 & \frac{\alpha_3}{\tilde{r}} + \beta_3 + \frac{\gamma_3 \tilde{z}}{\tilde{r}} & 0 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{pmatrix}$$

$$|A| = \frac{1}{2} \begin{vmatrix} 1 & r1 & z1 \\ 1 & r2 & z2 \\ 1 & r2 & z3 \end{vmatrix} = \frac{ab}{2}$$

$$[\tilde{B}] = \frac{1}{ab} \begin{pmatrix} -b & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & -a & 0 & a \\ b/2 & 0 & b/2 & 0 & b/2 & 0 \\ 0 & -b/2 & -a/2 & b/2 & a/2 & 0 \end{pmatrix}$$

$$\tilde{[B^T]}[E][\tilde{B}] = \frac{E}{a^2 b^2} \begin{pmatrix} 5b^2/4 & 0 & -3b^2/4 & 0 & b^2/4 & 0 \\ 0 & b^2/2 & ab/2 & -b^2/2 & -ab/2 & 0 \\ -3b^2/4 & ab/2 & (a^2/2) + (5b^2/4) & -ab/2 & (b^2/4) - (a^2/2) & 0 \\ 0 & -b^2/2 & -ab/2 & (a^2) + (b^2/2) & ab/2 & -a^2 \\ b^2/4 & -ab/2 & (b^2/4) - (a^2/2) & ab/2 & (a^2/2) + (b^2/4) & 0 \\ 0 & 0 & 0 & -a^2 & 0 & a^2 \end{pmatrix}$$

$$[K] = \frac{2\pi E}{3b} \begin{pmatrix} 5b^2/4 & 0 & -3b^2/4 & 0 & b^2/4 & 0 \\ 0 & b^2/2 & ab/2 & -b^2/2 & -ab/2 & 0 \\ -3b^2/4 & ab/2 & (a^2/2) + (5b^2/4) & -ab/2 & (b^2/4) - (a^2/2) & 0 \\ 0 & -b^2/2 & -ab/2 & (a^2) + (b^2/2) & ab/2 & -a^2 \\ b^2/4 & -ab/2 & (b^2/4) - (a^2/2) & ab/2 & (a^2/2) + (b^2/4) & 0 \\ 0 & 0 & 0 & -a^2 & 0 & a^2 \end{pmatrix}$$

2. Now, adding 1st, 3rd and 5th columns(or rows) produces a non-zero matrix. This signifies sum of displacement and force vector along radial direction is non-zero. Therefore, there is a non-equilibrium state.

$$\frac{2\pi E}{3b} \begin{pmatrix} 5b^2/4 \\ 0 \\ -3b^2/4 \\ 0 \\ b^2/4 \\ 0 \end{pmatrix} + \frac{2\pi E}{3b} \begin{pmatrix} -3b^2/4 \\ ab/2 \\ a^2/2 + 5b^2/4 \\ -ab/2 \\ b^2/4 - a^2/2 \\ 0 \end{pmatrix} + \frac{2\pi E}{3b} \begin{pmatrix} b^2/4 \\ -ab/2 \\ b^2/4 - a^2/2 \\ ab/2 \\ a^2/2 + b^2/4 \\ 0 \end{pmatrix} = \frac{\pi E}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Adding 2nd, 4th and 6th column (or rows) produces a zero matrix. There is a equilibrium in z- direction and sum of displacements and force vector along z-axis is zero.

$$\frac{2\pi E}{3b}\begin{pmatrix}0\\ b^2/2\\ ab/2\\ -b^2/2\\ -ab/2\\ 0\end{pmatrix}+\frac{2\pi E}{3b}\begin{pmatrix}0\\ -b^2/2\\ -ab/2\\ a^2+b^2/2\\ ab/2\\ -a^2\end{pmatrix}+\frac{2\pi E}{3b}\begin{pmatrix}0\\ 0\\ 0\\ -a^2\\ 0\\ a^2\end{pmatrix}=\frac{\pi E}{2}\begin{pmatrix}0\\0\\0\\0\\0\\0\end{pmatrix}$$

3.

$$N_1=\frac{\alpha_1+\beta_1 r+\gamma_1 z}{2A}=\frac{1}{3}\qquad N_2=\frac{\alpha_2+\beta_2 r+\gamma_2 z}{2A}=\frac{1}{3}\qquad N_3=\frac{\alpha_3+\beta_3 r+\gamma_3 z}{2A}=\frac{1}{3}$$

Using first approximation,

$$f^e=2\pi\int N^T[b]rdrdz=2\pi N^T[b]\tilde{r}A=2\pi\begin{pmatrix}1/3&0\\0&1/3\\1/3&0\\0&1/3\\1/3&0\\0&1/3\end{pmatrix}\begin{pmatrix}0\\-g\end{pmatrix}\frac{2a}{3}\frac{ab}{2}=\frac{2\pi a^2 b}{9}\begin{pmatrix}0\\-g\\0\\-g\\0\\-g\end{pmatrix}=\frac{2\pi a^2 bg}{9}\begin{pmatrix}0\\-1\\0\\-1\\0\\-1\end{pmatrix}$$