MASTER'S DEGREE NUMERICAL METHODS IN ENGINEERING

COMPUTATIONAL STRUCTURAL MECHANICS AND DYNAMICS

Assignment 5: Convergence requirements

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1 Assignment 5.1

The isoparametric definition of the straight–node bar element in its local system x is,

$$\begin{bmatrix} 1 \\ x \\ u \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} N_1^e(\xi) \\ N_2^e(\xi) \\ N_3^e(\xi) \end{bmatrix}$$

Here ξ is the isoparametric coordinate that takes the values –1, 1 and 0 at nodes 1, 2 and 3 respectively, while N_1^e, N_2^e and N_3^e are the shape functions for a bar element.

For simplicity, take $\bar{x_1} = 0, \bar{x_2} = L, \bar{x_3} = l/2 + l\alpha$. Here l is the bar length and α a parameter that characterizes how far node 3 is away from the midpoint location $\bar{x} = l/2$.

Show that the minimum α(minimal in absolute value sense) for which $J = \frac{d\bar{x}}{s\xi}$ vanishes at a point in the element are +_1/4 (the quarter points). Interpret this result as a singularity by showing that the axial strain becomes infinite at an end point.

1. In the first step we have to build up the shape functions for the quadratic 1D element in terms of ξ :

$$N_1 = \frac{(\xi_2 - \xi)(\xi_3 - \xi)}{(\xi_2 - \xi_1)(\xi_3 - \xi_1)}$$

$$N_2 = \frac{(\xi_3 - \xi)(\xi_1 - \xi)}{(\xi_3 - \xi_2)(\xi_1 - \xi_2)}$$

$$N_3 = \frac{(\xi_1 - \xi)(\xi_2 - \xi)}{(\xi_1 - \xi_3)(\xi_2 - \xi_3)}$$

Solving the N for $\xi_1 = -1, \xi_2 = 1, \xi_3 = 0$ we will have:

$$N_1 = \xi^2 - \xi/2, N_2 = \xi^2 + \xi/2, N_3 = 1 - \xi^2$$

We know that:

$$x = x_1 dN_1/d\xi + x_2 dN_2/d\xi + x_3 dN_3/d\xi$$

Using the given values for the x coordinates of the nodes in the physical world. The jacobian will be the sum of the derivatives of the shape functions in ξ

$$J = 0(\xi - 1/2) + l(\xi + 1/2) + (l/2 + l\alpha)(-2\xi)$$

$$J = \frac{l(1 - 4\alpha\xi)}{2}$$

In order for the jacobian to be zero and considering that $-1 < \xi < 1$ the only case the the term of $4\alpha\xi = 0$ is to $\alpha = -1/4, 1/4$ so that with $-1 < \xi < 1$ the term will be zero.

For calculating the axial strain we need to calculate the matrix "B" which is $B = J^{-1}[\frac{dN_1}{d\xi} \frac{dN_2}{d\xi} \frac{dN_3}{d\xi}]$. The axial strain is calculated as:

$$\epsilon = Bu = B \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

Because our jacobian is a one by one matrix then the inverse will be simply the inverse of the expression and matrix B can be calculated as for $\alpha = -1/4, 1/4$:

$$B = \begin{bmatrix} \frac{2}{l(1-(-1))}(\xi - 1/2) & \frac{2}{l(1-1)}(\xi + 1/2) & \frac{2}{l(1-0)}(-2\xi) \end{bmatrix}$$

The axial strain will be:

$$\epsilon = \begin{bmatrix} \frac{2}{l(1-(-1))}(\xi - 1/2) & \frac{2}{l(1-1)}(\xi + 1/2) & \frac{2}{l(1-0)}(-2\xi) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

Here in the second term of the B matrix we can see that the we have a term divided by zero which will cause our strain to explode.

2 Assignment 5.2

Extend the results obtained from the previous Exercise for a 9-node plane stress element. The element is initially a perfect square, nodes 5,6,7,8 are at the midpoint of the sides 1–2, 2–3, 3–4 and 4–1, respectively, and 9 at the center of the square.

Move node 5 tangentially towards 2 until the Jacobian determinant at 2 vanishes. This result is important in the construction of "singular elements" for fracture mechanics.

In the first step we have to calculate the shape functions of the 9 node quadrilateral element.

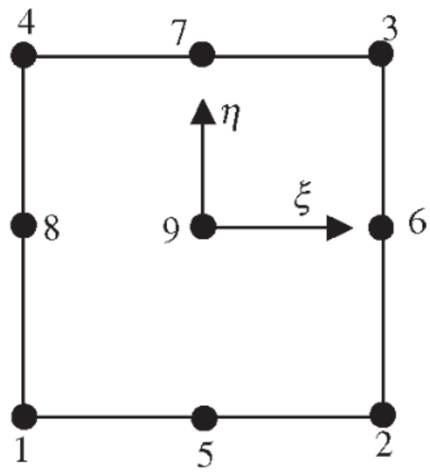

Figure 1: nine nod quadrilateral element

Calculating the shape functions we will have:

$$N_1 = \frac{\xi(\xi-1)\eta(\eta-1)}{4}, N_2 = \frac{(\xi+1)\xi\eta(\eta-1)}{4}$$

$$N_3 = \frac{(\xi+1)\xi\eta(\eta+1)}{4}, N_4 = \frac{(\xi-1)\xi\eta(\eta+1)}{4}$$

$$N_5 = \frac{(\xi+1)(\xi-1)\eta(\eta-1)}{-2}, N_6 = \frac{(\xi+1)\xi(\eta-1)(\eta+1)}{-2}$$

$$N_7 = \frac{(\xi+1)(\xi-1)\eta(\eta+1)}{-2}, N_8 = \frac{(\xi-1)\xi(\eta-1)(\eta+1)}{-2}$$

$$N_9 = \frac{(\xi+1)(\xi-1)(\eta-1)(\eta+1)}{1}$$

The jacobian can be calculated as:

$$J = \begin{bmatrix} x_i \Sigma \frac{dN_i}{d\xi} & y_i \Sigma \frac{dN_i}{d\xi} \\ x_i \Sigma \frac{dN_i}{d\eta} & y_i \Sigma \frac{dN_i}{d\eta} \end{bmatrix}$$

Calculating the jacobian with the derivatives of the shape functions we will have:

$$J(1,1) = x_1(2\xi-1)(\eta^2-\eta)/4 + x_2(2\xi+1)(\eta^2-\eta)/4 + x_3(2\xi+1)(\eta^2+\eta)/4 + x_4(2\xi-1)(\eta^2+\eta)/4$$
$$+x_5(-\xi)(\eta^2-\eta) + x_6(-2\xi-1)(\eta^2-1)/2 + x_7(-\xi)(\eta^2+\eta) + x_8(-2\xi+1)(\eta^2-1)/2 + x_9(2\xi)(\eta^2-1)$$

$$J(1,2) = y_1(2\xi-1)(\eta^2-\eta)/4 + y_2(2\xi+1)(\eta^2-\eta)/4 + y_3(2\xi+1)(\eta^2+\eta)/4 + y_4(2\xi-1)(\eta^2+\eta)/4$$
$$+y_5(-\xi)(\eta^2-\eta) + y_6(-2\xi-\xi)(\eta^2-1)/2 + y_7(-\xi)(\eta^2+\eta) + y_8(-2\xi+1)(\eta^2-1)/2 + y_9(2\xi)(\eta^2-1)$$

$$J(2,1) = x_1(2\eta-1)(\xi^2-\xi)/4 + x_2(2\eta-1)(\xi^2+\xi)/4 + x_3(2\eta+1)(\xi^2+\xi)/4 + x_4(2\eta+1)(\xi^2-\xi)/4$$
$$+x_5(-2\eta-1)(\xi^2-1)/2 + x_6(-\eta)(\xi^2+\xi) + x_7(-2\eta-1)(\xi^2-1)/2 + x_8(-\eta)(\xi^2-\xi)/2 + x_9(2\eta)(\xi^2-1)$$

$$J(2,1) = y_1(2\eta-1)(\xi^2-\xi)/4 + y_2(2\eta-1)(\xi^2+\xi)/4 + y_3(2\eta+1)(\xi^2+\xi)/4 + y_4(2\eta+1)(\xi^2-\xi)/4$$
$$+y_5(-2\eta-1)(\xi^2-1)/2 + y_6(-\eta)(\xi^2+\xi) + y_7(-2\eta-1)(\xi^2-1)/2 + y_8(-\eta)(\xi^2-\xi)/2 + y_9(2\eta)(\xi^2-1)$$

Considering the location of each point as:

$$x_1 = (0,0), x_2 = (l,0), x_3 = (l,l), x_4 = (0,l), x_5 = (l/2 + \alpha l, 0)$$
$$x_6 = (l, l/2), x_7 = (l/2, l), x_8 = (0, l/2), x_9 = (l/2, l/2)$$

We can see that the x and y coordinates for some of the nodes are the same. Simplifying the jacobian matrix for the coordinates we will have:

$$J(1,1) = \frac{2\xi+1}{2} x_2 + (-\xi)(\eta^2-\eta)x_5 + (\xi\eta^2 - \xi\eta - 2\xi)x_7$$

$$J(1,2) = 0$$

$$J(2,1) = \frac{-(\xi^2-1)(2\eta+1)}{2}x_5 + \frac{(\xi^2-1)(2\eta+1)}{2}x_7$$

$$J(2,2) = \frac{2\eta-1}{2}y_3 + 2\eta y_6$$

For the determinant of the jacobian at isoparametric coordinates of node 2 to be zero, as we can see because the J(1,2) is zero, for the value to be zero we need to put either J(1,1) or J(2,2) equal to zero and considering that we want to move node 5 towards node 2 until the determinant is zero we need to find the α in J(1,1) in a way the it is equal to zero. Considering $chi = 1, eta = -1$

$$\frac{2(1)+1}{2}l + (-(1))((-1)^2 - (-1))(l/2 + l\alpha) + ((1)(-1)^2 - (-1)(1) - 2(1))l/2 = 0$$

Solving the above equation for α we will find that for $\alpha = -1/4$, J(1,1) = 0 and the determinant of the jacobian will be equal to zero.