UNIVERSITAT POLITÈCNICA DE CATALUNYA

MASTER IN COMPUTATION MECHANICS AND NUMERICAL METHODS IN ENGINEERING

COMPUTATIONAL STRUCTURAL MECHANICS AND DYNAMICS

Assignment 5

by

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1- Introduction

The goal of the assignment is to apply the concept of convergence through different evaluations of the Jacobian Matrix **J** in different cases. A discussion on convergence requirements was also considered.

2 – Assignment 5.1

To obtain the Jacobian of the element (Figure 1) presented in the assignment [1], a relationship between the cartesian coordinate x and the natural coordinate ζ needs to be found. For such task, it is possible to parametrize x as a function of ζ with a second-degree polynomial since the presented element has 3 three nodes [1]. The second-degree polynomial is the following:

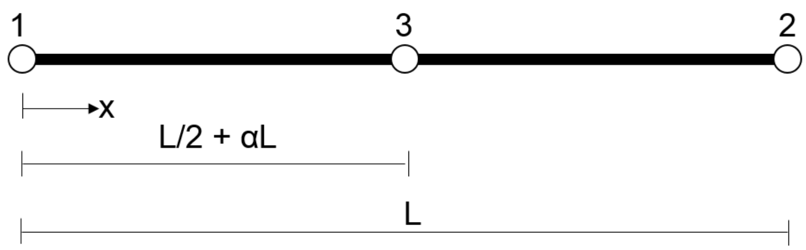

Figure 1. Element considered for Assignment 5.1.

$$x = \alpha_0 + \alpha_1 \zeta + \alpha_2 \zeta^2 \qquad (1)$$

Considering that when x = 0 → ζ = -1, Equation 1 can be rewritten as:

$$0 = \alpha_0 + -\alpha_1 + \alpha_2 \qquad (2)$$

Considering that when x = L → ζ = 1, Equation 1 can be rewritten a

$$L = \alpha_0 + \alpha_1 + \alpha_2 \qquad (3)$$

Considering that when x = L/2 + αL → ζ = 0, Equation 1 can be rewritten as

$$\mathrm{L}/2 + \alpha\mathrm{L} = \alpha_0 \qquad (4)$$

Equations 2-4 build a system which can be solved for the coefficients α_0, α_1 and α_2. The values of the coefficients are:

$$\alpha_0 = \mathrm{L}/2 + \alpha\mathrm{L}, \alpha_1 = L/2, \alpha_2 = -L\alpha$$

Replacing the values of the coefficients in Equation 1, the parametrization of x in terms of ζ becomes:

$$x = \frac{L}{2} + \alpha L + \frac{L}{2}\zeta - \alpha L\zeta^2 \tag{5}$$

With Equation (5), it is possible to calculate the Jacobian J:

$$J = \frac{dx}{d\zeta} = \frac{L}{2} - 2\alpha L\zeta \tag{6}$$

Considering the absolute value of +/- ¼ for α, Equation (6) takes the value of zero, therefore the jacobian J vanishes. Now, considering the following equation to evaluate the strains:

$$\varepsilon = \boldsymbol{Ba} \tag{7}$$

Where the **B** matrix is defined as:

$$\boldsymbol{B} = J^{-1}\frac{d\boldsymbol{N}}{d\zeta}, where\ \boldsymbol{N} = [N_1 N_2 N_3] \tag{8}$$

and N_i are the shape functions in natural coordinate ζ and are defined as [2]:

$$N_1 = \frac{\zeta^2}{2} - \frac{\zeta}{2}$$

$$N_2 = 1 + \zeta^2$$

$$N_3 = \frac{\zeta^2}{2} + \frac{\zeta}{2}$$

Calculating J^{-1} and the derivatives of the shape functions N_i w.r.t ζ, the **B** matrix is defined as:

$$\boldsymbol{B} = \left[\frac{2\zeta - 1}{L(1-4\alpha)}, \quad \frac{-4\zeta}{L(1-4\alpha)}, \quad \frac{1 + 2\zeta}{L(1-4\alpha)}\right]$$

According to the calculated **B** matrix, if α takes the absolute value of +/- ¼, the arguments of the B matrix become infinite, therefore the strains also become infinite (Equation 7).

3 – Assignment 5.2

Considering the 9-node quadrilateral element depicted below:

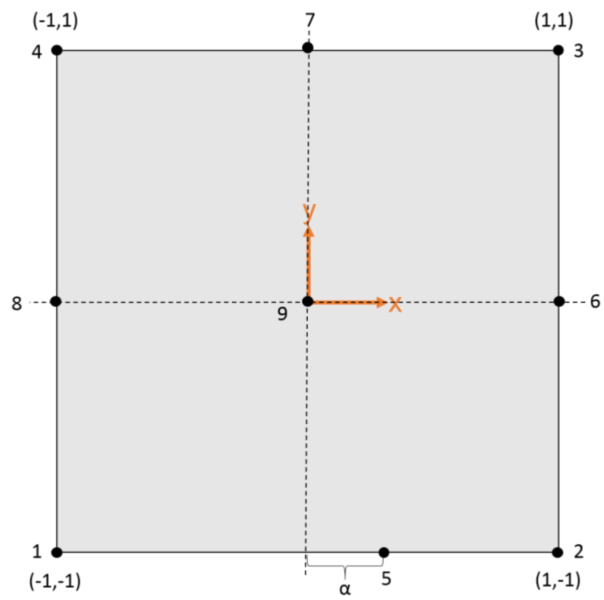

Figure 2. 9-node quadrilateral element in physical domain.

The parameter α in Figure 2 is the tangential shift of node 5 towards the node 2. The Jacobian Matrix **J** with respect to natural coordinates ζ and η and cartesian coordinates x and y (computational domain) can be evaluated through the following equation [2]:

$$\boldsymbol{J} = \sum_{i=1}^{9} \begin{bmatrix} \frac{\partial N_i}{\partial \zeta} x_i & \frac{\partial N_i}{\partial \zeta} y_i \\ \frac{\partial N_i}{\partial \eta} x_i & \frac{\partial N_i}{\partial \eta} y_i \end{bmatrix} \tag{9}$$

where x_i and y_i are the coordinates of the nodes in the element in the cartesian coordinates (Figure 2). N_i are the shape functions defined in natural coordinates w.r.t. to the element depicted in Figure 3.

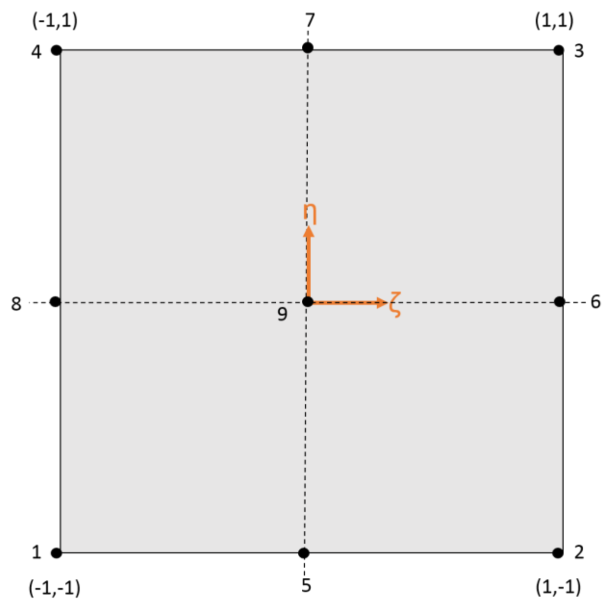

Figure 3. 9-node quadrilateral element in computational domain.

The N_i shape functions of the element depicted in Figure 3 are defined below [2]:

$$N_1 = \frac{1}{4}(1-\zeta)(1-\eta)\eta\zeta \tag{10.1}$$

$$N_2 = -\frac{1}{4}(1+\zeta)(1-\eta)\eta\zeta \tag{10.2}$$

$$N_3 = \frac{1}{4}(1+\zeta)(1+\eta)\eta\zeta \tag{10.3}$$

$$N_4 = -\frac{1}{4}(1-\zeta)(1+\eta)\eta\zeta \tag{10.4}$$

$$N_5 = -\frac{1}{2}(1-\zeta^2)(1-\eta)\eta \tag{10.5}$$

$$N_6 = \frac{1}{2}(1+\zeta)(1-\eta^2)\zeta \tag{10.6}$$

$$N_7 = \frac{1}{2}(1-\zeta^2)(1+\eta)\eta \tag{10.7}$$

$$N_8 = -\frac{1}{2}(1-\zeta)(1-\eta^2)\zeta \tag{10.8}$$

$$N_9 = (1-\zeta^2)(1-\eta^2) \tag{10.9}$$

Computing the derivatives of equations 10.1-10.9 with respect to ζ and η and considering the nodes' coordinates in cartesian coordinates (Figure 2), the Jacobian Matrix J (Equation 9) takes the following form:

$$\boldsymbol{J} = \begin{bmatrix} \alpha\eta\zeta(1-\eta)+1 & 0 \\ -\frac{1}{2}\alpha(1-\zeta^2)(1-2\eta) + 4\eta\zeta & 1 \end{bmatrix}$$

Computing the determinant of Jacobian Matrix **J** and setting it to zero:

$$\boldsymbol{det\,J} = 0 = \alpha\eta\zeta(1-\eta)+1$$

Solving it for α:

$$\alpha = -\frac{1}{\eta\zeta(1-\eta)} \tag{11}$$

Replacing the coordinates of node 2 (Figure 3) in Equation (11):

$$\alpha = \frac{1}{2}$$

Such value for α is in accordance with the result obtained in Section 2, because the current position of node 5 would be in the middle between node 2 and regular position of node 5 (α=0). The current position of node 5 (α = ½) would case the jacobian (determinant of Jacobian Matrix **J**) to be zero and would result in a singularity once the inverse of the Jacobian Matrix **J** is calculated.

4 – Discussion on Convergence Requirements

The evaluation of the jacobian is a measure to determine whether the discretization of the initial domain was performed in a plausible manner or not. As presented in the sections above, the positioning of the nodes is critical to enable the computation of the stiffness matrix **K** of an element and avoid singularities. Such observation has great importance when mesh convergence is under evaluation. If the elements lack minimum quality, the jacobian can be zero or even negative. In such cases, the stability of the solution is lost, and the convergence of the solution is affected. Consistency and stability are necessary for the convergence of the solution and some aspects are critical to evaluate both features. Two necessary requirements to enable convergence to the correct solution are presented in the following.

If the sum of shape functions inside an element is different than one, the requirement of completeness is not met and the solution may not converge or even converge to an erroneous solution. The lack of completeness deteriorates the consistency of the solution and convergence is precluded [2, 3]. Also, if the element does not present geometrical invariance, the solution cannot converge. The geometrical invariance can be obtained by applying the same shape functions to all degrees of freedom of the element [2]. It is worth mentioning that both requirements are related to shape functions, which enforces the importance of choosing suitable shape functions to solve the problem. Other features regarding stability (rank sufficiency) and consistency (compatibility) of the solution should also be evaluated to aid the convergence of the solution [4].

5 – References

[1] – Assignment-5, Computational Structural Mechanics and Dynamics, Master of Science in Computational Mechanics, 2020.

[2] – Oñate, E., “Structural Analysis with the Finite Element Method – Linear Statics”, vol. 1, 2008.

[3] – “Variational Crimes and the Patch Test”, Computational Structural Mechanics and Dynamics, Master of Science in Computational Mechanics, 2020.

[4] – Presentation “CSMD-09-Convergence”, Computational Structural Mechanics and Dynamics, Master of Science in Computational Mechanics, 2020.