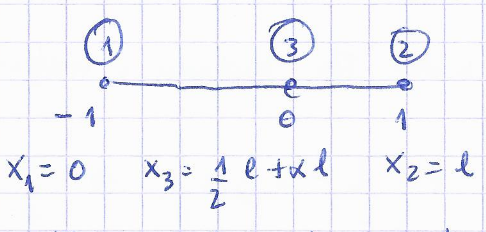
Computational Structural Mechanics and Dynamics — Aitor Bazán Escoda

On "Convergence requirements" — 16/03/2020

Assignment 5.1

Isoparametric definition of straight-node bar element in its local system $\bar{x}$

$$\begin{bmatrix} 1 \\ \bar{x} \\ \bar{u} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ \bar{x}_1 & \bar{x}_2 & \bar{x}_3 \\ \bar{u}_1 & \bar{u}_2 & \bar{u}_3 \end{bmatrix} \begin{bmatrix} N_1^e(\xi) \\ N_2^e(\xi) \\ N_3^e(\xi) \end{bmatrix}$$



- Show that the minimum α for which $J = \frac{d\bar{x}}{d\xi}$ vanishes at a point in the element are $\pm 1/4$ (quarter points). Interpret the result as a singularity by showing that the axial strain becomes infinite at an end point.

First, the quadratic shape functions for that element are calculated:

$$N_1^e(\xi) = \frac{1}{2}\xi(\xi - 1)$$

$$N_2^e(\xi) = \frac{1}{2}\xi(\xi + 1)$$

$$N_3^e(\xi) = -(\xi+1)(\xi-1) = -(\xi^2 - 1)$$

now, when isoparametric formulation is used, the mapping geometry yields as,

$$\bar{x} = \bar{x}_1 N_1^e(\xi) + \bar{x}_2 N_2^e(\xi) + \bar{x}_3 N_3^e(\xi)$$

$$\bar{x} = 0\, N_1^e(\xi) + \frac{1}{2} l \xi(\xi+1) - \left(\frac{1}{2}l + \alpha l\right)(\xi^2 - 1)$$

and so,

$$\bar{x} = \frac{1}{2} l \xi(\xi+1) - (\xi^2 - 1)\left(\frac{1}{2}l + \alpha l\right)$$

Since the Jacobian is defined as:

$$J := \frac{d\bar{x}}{d\xi} = \bar{x}_1 \frac{dN_1^e(\xi)}{d\xi} + \bar{x}_2 \frac{dN_2^e(\xi)}{d\xi} + \bar{x}_3 \frac{dN_3^e(\xi)}{d\xi}$$

thus,

$$J = \bar{x}_1\left(\xi - \frac{1}{2}\right) + \bar{x}_2\left(\xi + \frac{1}{2}\right) - 2\xi\bar{x}_3$$

if the values for nodal coordinates are now substituted,

$$J = \ell\left(\xi+\frac{1}{2}\right) - 2\xi\left(\frac{1}{2}\ell+\alpha\ell\right) = \ell\left(\xi+\frac{1}{2} - 2\xi\left(\frac{1}{2}\ell+\alpha\ell\right)\right) =$$

$$= \ell\left(\frac{1}{2} - 2\xi\alpha\right)$$

Therefore, when the Jacobian vanishes, it implies $J=0$

$$0 = \ell\left(\frac{1}{2} - 2\xi\alpha\right) \Rightarrow \frac{1}{2} = 2\xi\alpha \Rightarrow \xi = \frac{1}{4\alpha}$$

The Jacobian vanishes at that point of the element bar.

but, we want the critical value obtained above, to be within the element domain $\Rightarrow \xi = [-1, 1]$

$$-1 \le \xi \le 1 \Rightarrow -1 \le \frac{1}{4\alpha} \le 1 \Rightarrow -\frac{1}{4} \le \alpha \le \frac{1}{4}$$

and so, $|\alpha| = \frac{1}{4}$ when $\xi = \pm 1$; thus, the Jacobian vanishes at one of these two end points.

Now, since we know that, $\bar{u} = \bar{u}_1 N_1^e(\xi) + \bar{u}_2 N_2^e(\xi) + \bar{u}_3 N_3^e(\xi)$

and, from the given definition,

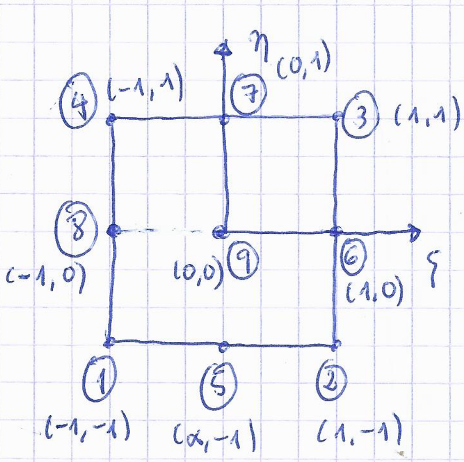
$$\varepsilon = \frac{d\bar{u}}{d\bar{x}} \Rightarrow \varepsilon = \bar{u}_1 \frac{dN_1^e}{d\bar{x}} + \bar{u}_2 \frac{dN_2^e}{d\bar{x}} + \bar{u}_3 \frac{dN_3^e}{d\bar{x}}$$

By using the chain-rule,

$$\frac{dN_i}{d\bar{x}} = \frac{dN_i}{d\xi}\frac{d\xi}{d\bar{x}} = \frac{dN_i}{d\xi} J^{-1}$$

and so, here it is shown that the strain is a function of the inverse of the Jacobian. When the Jacobian is null, the strain tends to infinity.

Assignment 5.2



Nodal coordinates are:

$x_1 = -1$; $x_2 = 1$; $x_3 = 1$; $x_4 = -1$; $x_5 = \alpha$; $x_6 = 1$

$x_7 = 0$; $x_8 = -1$; $x_9 = 0$

$y_1 = -1$; $y_2 = -1$; $y_3 = 1$; $y_4 = 1$; $y_5 = -1$

$y_6 = 0$; $y_7 = 1$; $y_8 = 0$; $y_9 = 0$

Element shape functions for the current 9-node quadratic element (Q9) are:

$$N_1^e(\xi,\eta) = \frac{1}{4}(\xi-1)(\eta-1)\xi\eta$$

$$N_2^e(\xi,\eta) = \frac{1}{4}(\xi+1)(\eta-1)\xi\eta$$

$$N_3^e(\xi,\eta) = \frac{1}{4}(\xi+1)(\eta+1)\xi\eta$$

$$N_4^e(\xi,\eta) = \frac{1}{4}(\xi-1)(\eta+1)\xi\eta$$

$$N_5^e(\xi,\eta) = \frac{1}{2}(1-\xi^2)\eta(\eta-1)$$

$$N_6^e(\xi,\eta) = \frac{1}{2}(1-\eta^2)\xi(\xi+1)$$

$$N_7^e(\xi,\eta) = \frac{1}{2}(1-\xi^2)\eta(\eta+1)$$

$$N_8^e(\xi,\eta) = \frac{1}{2}(1-\eta^2)\xi(\xi-1)$$

$$N_9^e(\xi,\eta) = (1-\xi^2)(1-\eta^2)$$

the relation between the global coordinates and the isoparametric coordinates is made through;

$$\bar{x} = \sum_{i=1}^{9} \bar{x}_i N_i \qquad \bar{y} = \sum_{i=1}^{9} \bar{y}_i N_i$$

Thus,

$$\begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 & -1 & \alpha & 1 & 0 & -1 & 0 \\ -1 & -1 & 1 & 1 & -1 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} N_1^e(\xi,\eta) \\ N_2^e(\xi,\eta) \\ N_3^e(\xi,\eta) \\ N_4^e(\xi,\eta) \\ N_5^e(\xi,\eta) \\ N_6^e(\xi,\eta) \\ N_7^e(\xi,\eta) \\ N_8^e(\xi,\eta) \\ N_9^e(\xi,\eta) \end{bmatrix}$$

$$\bar{x} = \sum_{i=1}^{9} \bar{x}_i N_i^e = -N_1^e(\xi,\eta) + N_2^e(\xi,\eta) + N_3^e(\xi,\eta) - N_4^e(\xi,\eta) + \alpha N_5^e(\xi,\eta) + N_6^e(\xi,\eta) - N_8^e(\xi,\eta)$$

$$\bar{x} = \frac{1}{2}\xi(\eta^2-1)(\xi-1) - \frac{1}{2}\xi(\eta^2-1)(\xi+1) - \frac{1}{4}\xi\eta(\xi-1)(\eta-1) - \frac{1}{4}\xi\eta(\xi-1)(\eta+1) + \frac{1}{4}\xi\eta(\xi+1)(\eta-1) + \frac{1}{4}\xi\eta(\xi+1)(\eta+1) - \frac{1}{2}\alpha\eta(\xi^2-1)(\eta-1)$$

$$\bar{y} = \sum_{i=1}^{9} \bar{y}_i N_i^e = -N_1^e(\xi,\eta) - N_2^e(\xi,\eta) + N_3^e(\xi,\eta) + N_4^e(\xi,\eta) - N_5^e(\xi,\eta) + N_7^e(\xi,\eta)$$

$$\bar{y} = \frac{1}{2}\eta(\xi^2-1)(\eta-1) - \frac{1}{2}\eta(\xi^2-1)(\eta+1) - \frac{1}{4}\xi\eta(\xi-1)(\eta-1) + \frac{1}{4}\xi\eta(\xi-1)(\eta+1) - \frac{1}{4}\xi\eta(\xi+1)(\eta-1) + \frac{1}{4}\xi\eta(\xi+1)(\eta+1)$$

now, since we now the Jacobian is defined as,

$$J(\xi,\eta) = \begin{bmatrix} \frac{d\bar{x}}{d\xi} & \frac{d\bar{y}}{d\xi} \\ \frac{d\bar{x}}{d\eta} & \frac{d\bar{y}}{d\eta} \end{bmatrix} = \sum_{i=1}^{9} \begin{bmatrix} \bar{x}_i \frac{dN_i^e}{d\xi} & \bar{y}_i \frac{dN_i^e}{d\xi} \\ \bar{x}_i \frac{dN_i^e}{d\eta} & \bar{y}_i \frac{dN_i^e}{d\eta} \end{bmatrix}$$

each one of the terms involving the Jacobian definition read as:

$$J_{11}(\xi,\eta) = \frac{d\bar{x}}{d\xi} = \frac{1}{2}\xi\eta(\xi+1) - \frac{1}{2}\xi\eta(\xi-1) - \frac{1}{2}\alpha(\xi^2-1)(\eta-1)$$

$$-\frac{1}{2}\alpha\eta(\xi^2-1) - \frac{1}{4}\xi(\xi-1)(\eta-1) - \frac{1}{4}\xi(\xi-1)(\eta+1) + \frac{1}{4}\xi(\xi+1)(\eta-1)$$

$$+\frac{1}{4}\xi(\xi+1)(\eta+1) + \xi\eta(\xi-1) - \xi\eta(\xi+1)$$

$$J_{12}(\xi,\eta) = \frac{d\bar{y}}{d\xi} = \frac{1}{2}(\xi^2-1)(\eta-1) - \frac{1}{2}(\xi^2-1)(\eta+1) - \frac{1}{4}\xi(\xi-1)(\eta-1)$$

$$+\frac{1}{4}\xi(\xi-1)(\eta+1) - \frac{1}{4}\xi(\xi+1)(\eta-1) + \frac{1}{4}\xi(\xi+1)(\eta+1)$$

$$J_{21}(\xi,\eta) = \frac{d\bar{x}}{d\eta} = \frac{1}{2}\xi\eta(\xi+1) - \frac{1}{2}\xi\eta(\xi-1) - \frac{1}{2}\alpha(\xi^2-1)(\eta-1)$$

$$-\frac{1}{2}\alpha\eta(\xi^2-1) - \frac{1}{4}\xi(\xi-1)(\eta-1) - \frac{1}{4}\xi(\xi-1)(\eta+1) + \frac{1}{4}\xi(\xi+1)(\eta-1)$$

$$+\frac{1}{4}\xi(\xi+1)(\eta+1) + \xi\eta(\xi-1) - \xi\eta(\xi+1)$$

$$J_{22}(\xi,\eta) = \frac{d\bar{y}}{d\eta} = \frac{1}{2}(\xi^2-1)(\eta-1) - \frac{1}{2}(\xi^2-1)(\eta+1) - \frac{1}{4}\xi(\xi-1)(\eta-1)$$

$$+\frac{1}{4}\xi(\xi-1)(\eta+1) - \frac{1}{4}\xi(\xi+1)(\eta-1) + \frac{1}{4}\xi(\xi+1)(\eta+1)$$

therefore, evaluating the Jacobian at node 2, this is at element coordinates $(\xi,\eta) = (1,-1)$, ~~then~~

$$J_{11}(\xi,\eta)\big|_{(1,-1)} = \frac{d\bar{x}}{d\xi}\Big|_{(1,-1)} = 1-2\alpha$$

$$J_{12}(\xi,\eta)\big|_{(1,-1)} = \frac{d\bar{y}}{d\xi}\Big|_{(1,-1)} = 0$$

$$J_{21}(\xi,\eta)\big|_{(1,-1)} = \frac{d\bar{x}}{d\eta}\Big|_{(1,-1)} = 0$$

$$J_{22}(\xi,\eta)\big|_{(1,-1)} = \frac{d\bar{y}}{d\eta}\Big|_{(1,-1)} = 1$$

Finally, the Jacobian can be expressed as $J(1,-1) = \begin{bmatrix} 1-2\alpha & 0 \\ 0 & 1 \end{bmatrix}$

The condition to make the Jacobian vanish is when its determinant is null (2D case)

$$\left| J(\xi,\eta)\big|_{(1,-1)} \right| = \begin{vmatrix} 1-2\alpha & 0 \\ 0 & 1 \end{vmatrix} = 0 \Rightarrow 1-2\alpha = 0 \quad \boxed{\alpha = \frac{1}{2}}$$

Therefore, the condition for the Jacobian to vanish is when $\alpha = \frac{1}{2}$ (quarter point), similarly to the 1D case condition obtained in the first case (Assignment 5.1)