Universitat Politècnica De Catalunya, Barcelona

MSc. Computational Mechanics Erasmus Mundus

Assignment 5: Isoparametric representation

Computational Structural Mechanics & Dynamics

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On "Isoparametric representation":

Problem 5.1

Consider a three-node bar element referred to the natural coordinate ξ. The two end nodes and the mid node are identified as 1, 2 and 3 respectively. The natural coordinates of nodes 1, 2 and 3 are $\xi = -1$, $\xi = 1$ and $\xi = 0$, respectively. The variation of the shape functions $N_1(\xi)$, $N_2(\xi)$ and $N_3(\xi)$ is sketched in the figure below. These functions must be quadratic polynomials in ξ:

$$N_1^e(\xi) = a_0 + a_1\xi + a_2\xi^2 \quad N_2^e(\xi) = b_0 + b_1\xi + b_2\xi^2 \quad N_3^e(\xi) = c_0 + c_1\xi + c_2\xi^2$$

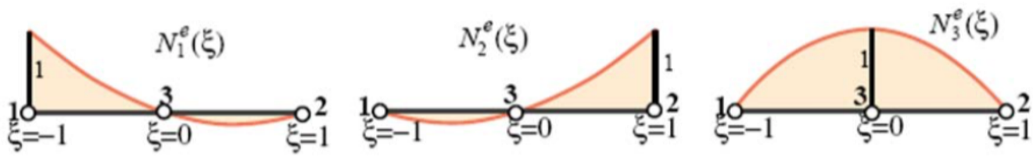

Figure 1: Isoparametric shape functions for 3-node bar element (sketch). Node 3 has been drawn at the 1-2 midpoint but it may be moved away from it.

a) Determine the coefficients $a_0, ..., c_2$ using the node value conditions depicted in the figure. For example $N_1^e = 1$ for $\xi = -1$ and 0 for the rest of natural coordinates. The rest of the nodes follow the same scheme.

Solution: Let us consider the first shape function N_1^e. From figure 1, we can write the node values at each natural coordinate. For example,

$$\text{For } \xi = -1, N_1^e(\xi) = 1 \quad \Longrightarrow \quad N_1^e(-1) = a_0 + a_1(-1) + a_2(-1)^2 = 1$$

$$a_0 - a_1 + a_2 = 1 \tag{1}$$

$$\text{For } \xi = 0, N_1^e(\xi) = 0 \quad \Longrightarrow \quad N_1^e(0) = a_0 + a_1(0) + a_2(0)^2 = 0$$

$$a_0 = 0 \tag{2}$$

$$\text{For } \xi = 1, N_1^e(\xi) = 0 \quad \Longrightarrow \quad N_1^e(1) = a_0 + a_1(1) + a_2(1)^2 = 0$$

$$a_0 + a_1 + a_2 = 0 \tag{3}$$

From equations (1), (2) and (3), we get,

$$\boxed{a_0 = 0, \quad a_1 = -\frac{1}{2}, \quad a_2 = \frac{1}{2}}$$

Thus we have,

$$N_1^e(\xi) = -\frac{1}{2}\xi(1-\xi)$$

Similarly, for the second shape function we can write,

$$\text{For } \xi = -1, N_2^e(\xi) = 0 \quad \Longrightarrow N_2^e(-1) = b_0 + b_1(-1) + b_2(-1)^2 = 0$$

$$b_0 - b_1 + b_2 = 0 \tag{4}$$

$$\text{For } \xi = 0, N_2^e(\xi) = 0 \quad \Longrightarrow N_2^e(0) = b_0 + b_1(0) + b_2(0)^2 = 0$$

$$b_0 = 0 \tag{5}$$

$$\text{For } \xi = 1, N_2^e(\xi) = 1 \quad \Longrightarrow N_2^e(1) = b_0 + b_1(1) + b_2(1)^2 = 1$$

$$b_0 + b_1 + b_2 = 1 \tag{6}$$

From equations (4), (5) and (6), we get,

$$\boxed{b_0 = 0, \quad b_1 = \frac{1}{2}, \quad b_2 = \frac{1}{2}}$$

Thus we have,

$$N_2^e(\xi) = \frac{1}{2}\xi(1+\xi)$$

Lastly, for the third shape function, we repeat the same process,

$$\text{For } \xi = -1, N_3^e(\xi) = 0 \quad \Longrightarrow N_3^e(-1) = c_0 + c_1(-1) + c_2(-1)^2 = 0$$

$$c_0 - c_1 + c_2 = 0 \tag{7}$$

$$\text{For } \xi = 0, N_3^e(\xi) = 1 \quad \Longrightarrow N_3^e(0) = c_0 + c_1(0) + c_2(0)^2 = 1$$

$$c_0 = 1 \tag{8}$$

$$\text{For } \xi = 1, N_3^e(\xi) = 0 \quad \Longrightarrow N_3^e(1) = c_0 + c_1(1) + c_2(1)^2 = 0$$

$$c_0 + c_1 + c_2 = 0 \tag{9}$$

From equations (7), (8) and (9), we get,

$$\boxed{c_0 = 1, \quad c_1 = 0, \quad c_2 = -1}$$

Thus we have,

$$N_3^e(\xi) = 1 - \xi^2$$

Therefore, we get the shape functions as,

$$\boxed{N_1^e(\xi) = -\frac{1}{2}\xi(1-\xi), \quad N_2^e(\xi) = \frac{1}{2}\xi(1+\xi), \quad N_3^e(\xi) = 1 - \xi^2}$$

b) Verify that their sum is identically one.

Solution: The shape functions derived in the last section are given as,

$$N_1^e(\xi) = -\frac{1}{2}\xi(1-\xi), \quad N_2^e(\xi) = \frac{1}{2}\xi(1+\xi), \quad N_3^e(\xi) = 1-\xi^2$$

It is clearly seen that the sum of the shape functions is unity i.e.

$$-\frac{1}{2}\xi + \frac{1}{2}\xi^2 + \frac{1}{2}\xi + \frac{1}{2}\xi^2 + 1 - \xi^2 = 1$$

Therefore it is verified,

$$\boxed{N_1^e(\xi) + N_2^e(\xi) + N_3^e(\xi) = 1}$$

c) Calculate their derivatives respect to the natural coordinates.

Solution: Now, we calculate the derivatives of the shape functions with respect to the natural coordinates,

$$\frac{dN_1^e(\xi)}{d\xi} = \frac{d(-\frac{1}{2}\xi + \frac{1}{2}\xi^2)}{d\xi} \implies \boxed{\frac{dN_1^e(\xi)}{d\xi} = -\frac{1}{2} + \xi}$$

$$\frac{dN_2^e(\xi)}{d\xi} = \frac{d(\frac{1}{2}\xi + \frac{1}{2}\xi^2)}{d\xi} \implies \boxed{\frac{dN_2^e(\xi)}{d\xi} = \frac{1}{2} + \xi}$$

$$\frac{dN_3^e(\xi)}{d\xi} = \frac{d(1-\xi^2)}{d\xi} \implies \boxed{\frac{dN_3^e(\xi)}{d\xi} = -2\xi}$$

It is interesting to note that the sum of the derivatives of the shape functions is also unity, i.e.

$$\frac{dN_1^e(\xi)}{d\xi} + \frac{dN_2^e(\xi)}{d\xi} + \frac{dN_3^e(\xi)}{d\xi} = 1$$

Problem 5.2

A five node quadrilateral element has the nodal configuration shown in the figure with two perspective views of N_1^e and N_5^e. Find five shape functions $N_i^e, i = 1, ..., 5$ that satisfy compatibility and also verify that their sum is unity.

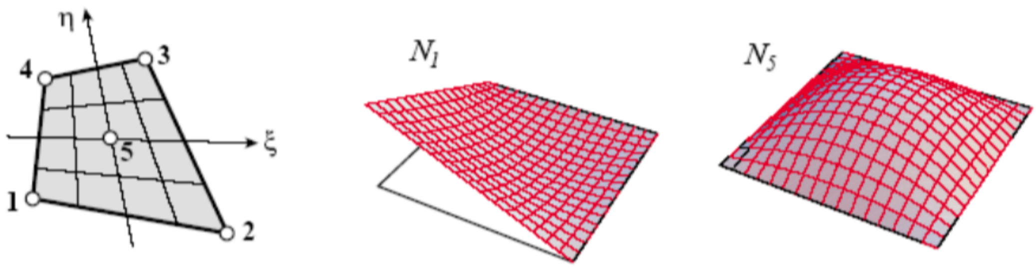

Figure 2

Solution: Firstly, we use the line-product method for finding the shape function for node 5. From figure 2 we can examine that,

$$N_5^e = c_1 \, L_{1-2} \, L_{2-3} \, L_{3-4} \, L_{4-1} \tag{10}$$

It is by construction that this expression vanishes over the nodes 1, 2, 3, 4 (or over the sides 1-2, 2-3, 3-4 and 4-1) and can be normalised to unity at node 5 by adjusting the value of c_1. The equation for the four sides, 1-2, 2-3, 3-4 and 4-1, are known to be $\eta = -1, \xi = 1, \eta = 1$ and $\xi = -1$ respectively. Using these in equation (10), we get,

$$N_5^e(\xi, \eta) = c_1 \, (\eta + 1) \, (\eta - 1) \, (\xi + 1) \, (\xi - 1)$$

We evaluate this expression at node 5, to find the value of c_1, which has natural coordinates of $\xi = \eta = 0$.

$$N_5^e(0, 0) = c_1 \, (1) \, (-1) \, (1) \, (-1) = 1$$

Thus, $c_1 = 1$ and shape function for node 5 is,

$$\boxed{N_5^e(\xi, \eta) = (\eta + 1) \, (\eta - 1) \, (\xi + 1) \, (\xi - 1)}$$

Now, we know the corner shape functions $\bar{N}_i^e(\xi, \eta)$ with $i = 1, 2, 3, 4$ for the 4-node quadrilateral are given as,

$$\bar{N}_1^e = \frac{1}{4}(1 - \eta) \, (1 - \xi)$$

$$\bar{N}_2^e = \frac{1}{4}(1 - \eta) \, (1 + \xi)$$

$$\bar{N}_3^e = \frac{1}{4}(1 + \eta) \, (1 + \xi)$$

$$\bar{N}_4^e = \frac{1}{4}(1 + \eta) \, (1 - \xi)$$

Despite the fact that the shape functions of the corner nodes resembles the shape functions of a 4-node quadrilateral, they are not the same i.e. $N_i \neq \bar{N}_i$. For example, the corner shape function N_1^e shown in figure 2 must vanish at node 5 (with $\xi = \eta = 0$). However it takes a value of 1/4. This is illustrated in table 1.

Node	1	2	3	4	5
N_1^e	1	0	0	0	$\frac{1}{4}$
N_2^e	0	1	0	0	$\frac{1}{4}$
N_3^e	0	0	1	0	$\frac{1}{4}$
N_4^e	0	0	0	1	$\frac{1}{4}$
N_5^e	0	0	0	0	1

Table 1: Corner shape functions not vanishing at node 5

In order to combat this situation, we define a factor α in the expression for the corner shape functions so that all corner N_i^e (for i = 1, 2, 3, 4) vanish at node 5. Thus,

$$N_i^e = \bar{N}_i^e + \alpha\, N_5^e \quad \text{for} \quad i = 1, 2, 3, 4 \tag{11}$$

The use of the factor α in the corner shape functions is shown in table 2, where it is important to note that for the corner shape functions to vanish at node 5,

$$\frac{1}{4} + \alpha = 0 \qquad \Longrightarrow \boxed{\alpha = -\frac{1}{4}}$$

Node	1	2	3	4	5
N_1^e	1	0	0	0	$\frac{1}{4} + \alpha$
N_2^e	0	1	0	0	$\frac{1}{4} + \alpha$
N_3^e	0	0	1	0	$\frac{1}{4} + \alpha$
N_4^e	0	0	0	1	$\frac{1}{4} + \alpha$
N_5^e	0	0	0	0	1

Table 2: Corner shape functions using the factor α which vanishes them for $\alpha = -\frac{1}{4}$

Therefore, using $\alpha = -\frac{1}{4}$ in the equation (11), we get the shape functions of a 5-node quadrilateral element that satisfy compatibility,

$$
\boxed{
\begin{aligned}
N_1^e &= \frac{1}{4}(1-\eta)\,(1-\xi) - \frac{1}{4}\,(\eta+1)\,(\eta-1)\,(\xi+1)\,(\xi-1) && (12)\\
N_2^e &= \frac{1}{4}(1-\eta)\,(1+\xi) - \frac{1}{4}\,(\eta+1)\,(\eta-1)\,(\xi+1)\,(\xi-1) && (13)\\
N_3^e &= \frac{1}{4}(1+\eta)\,(1+\xi) - \frac{1}{4}\,(\eta+1)\,(\eta-1)\,(\xi+1)\,(\xi-1) && (14)\\
N_4^e &= \frac{1}{4}(1+\eta)\,(1-\xi) - \frac{1}{4}\,(\eta+1)\,(\eta-1)\,(\xi+1)\,(\xi-1) && (15)\\
N_5^e &= (\eta+1)\,(\eta-1)\,(\xi+1)\,(\xi-1) && (16)
\end{aligned}
}
$$

Also, it is verified that the sum of the shape functions is unity i.e.

$$\sum_{i=1}^{5} N_i^e = 1$$

Problem 5.3

On "Convergence requirements":

Which minimum integration rules of Gauss-product type gives a rank sufficient stiffness matrix for these elements:

1. the 8-node hexahedron
2. the 20-node hexahedron
3. the 27-node hexahedron
4. the 64-node hexahedron

Solution: Let us assume that the Gaussian formula is used with stress-strain matrix $\boldsymbol{E}$ constant over the element. Then the numerical integration of the stiffness matrix is given by,

$$\boldsymbol{K}^e = \sum_{i=1}^{p_1}\sum_{j=1}^{p_2}\sum_{k=1}^{p_3} w_{ijk}\,\boldsymbol{B}_{ijk}^T\,\boldsymbol{E}\,\boldsymbol{B}_{ijk}\,J_{ijk} \tag{17}$$

where the number of Gauss points along the ξ, η and ν directions is denoted by p_1, p_2 and p_3 respectively.

The rule is identified as $p_1 \times p_2 \times p_3$ and has $p_1 p_2 p_3$ points. For conventional hexahedral elements, the number of integration points is taken the same in all directions i.e. $p = p_1 = p_2 = p_3$, and the total number of Gauss points is $n_G = p^3$, This is known as the isotropic product rule and each point adds 6 to the stiffness matrix rank.

1. For a 8-node hexahedron (n = 8),

All degree of freedom (dofs) = 8*3 = 24
Subtracting the rigid body modes, for a rank stiffness matrix, we get 24 - 6 = 18
Therefore the condition for a 8-node hexahedron is given by,

$$\boxed{6n_G = 6p^3 \geq 18 \quad \Longrightarrow \quad n_G \geq 3 \quad \Longrightarrow \quad \text{Rank sufficient for} \quad 2 \times 2 \times 2}$$

Therefore the 8-point product rule gives a rank sufficient stiffness matrix $\boldsymbol{K}^e$ for a 8-node hexahedron.

2. For a 20-node hexahedron (n = 20),

All degree of freedom (dofs) = 20*3 = 60
Subtracting the rigid body modes, for a rank stiffness matrix, we get 60 - 6 = 54
Therefore the condition for a 20-node hexahedron is given by,

$$\boxed{6n_G = 6p^3 \geq 54 \quad \Longrightarrow \quad n_G \geq 9 \quad \Longrightarrow \quad \text{Rank sufficient for} \quad 3 \times 3 \times 3}$$

Therefore the 27-point product rule gives a rank sufficient stiffness matrix $\boldsymbol{K}^e$ for a 20-node hexahedron.

3. For a 27-node hexahedron (n = 27),

All degree of freedom (dofs) = 27*3 = 81
Subtracting the rigid body modes, for a rank stiffness matrix, we get 81 - 6 = 75
Therefore the condition for a 27-node hexahedron is given by,

$$\boxed{6n_G = 6p^3 \geq 75 \quad \Longrightarrow \quad n_G \geq 12.5 \quad \Longrightarrow \quad \text{Rank sufficient for} \quad 3 \times 3 \times 3}$$

Therefore the 27-point product rule gives a rank sufficient stiffness matrix $\boldsymbol{K}^e$ for a 27-node hexahedron.

4. For a 64-node hexahedron (n = 64),

All degree of freedom (dofs) = 64*3 = 192
Subtracting the rigid body modes, for a rank stiffness matrix, we get 192 - 6 = 186
Therefore the condition for a 64-node hexahedron is given by,

$$\boxed{6n_G = 6p^3 \geq 186 \quad \Longrightarrow \quad n_G \geq 31 \quad \Longrightarrow \quad \text{Rank sufficient for} \quad 4 \times 4 \times 4}$$

Therefore the 64-point product rule gives a rank sufficient stiffness matrix $\boldsymbol{K}^e$ for a 64-node hexahedron.