Computational Structural Mechanics and Dynamics

Master of Science in Computational Mechanics
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Homework 5

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Problem 5.1

Consider a three-node bar element referred to the natural coordinate ξ. The two end nodes and the mid node are identified as 1, 2 and 3 respectively. The natural coordinates of nodes 1, 2 and 3 are $\xi = -1$, $\xi = 1$ and $\xi = 0$, respectively. The variation of the shape functions $N_1(\xi)$, $N_2(\xi)$ and $N_3(\xi)$ is sketched in the figure below. These functions must be quadratic polynomials in ξ:

$$N_1^e(\xi) = a_0 + a_1\xi + a_2\xi^2 \qquad N_2^e(\xi) = b_0 + b_1\xi + b_2\xi^2 \qquad N_3^e(\xi) = c_0 + c_1\xi + c_2\xi^2$$

(a) Determine the coefficients $a0, ..., c2$ using the node value conditions depicted in figure. For example $N_1^e = 1$ for $\xi = 1$ and 0 for the rest of natural coordinates. The rest of the nodes follow the same scheme.

$$N_1^e(-1) = a_0 - a_1 + a_2 = 1 \qquad N_2^e(-1) = b_0 - b_1 + b_2 = 0 \qquad N_3^e(-1) = c_0 - c_1 + c_2 = 0$$

$$N_1^e(0) = a_0 = 0 \qquad N_2^e(0) = b_0 = 0 \qquad N_3^e(0) = c_0 = 1$$

$$N_1^e(+1) = a_0 + a_1 + a_2 = 0 \qquad N_2^e(+1) = b_0 + b_1 + b_2 = 1 \qquad N_3^e(+1) = c_0 + c_1 + c_2 = 0$$

Solving the linear system we can know the values for $a0, ..., c2$:

$$N_1^e(\xi) = \frac{1}{2}\xi(1-\xi) \qquad N_2^e(\xi) = \frac{1}{2}\xi(1+\xi) \qquad N_3^e(\xi) = 1 - \xi^2$$

(b) Verify that their sum is identically one.

$$N_1 + N_2 + N_3 = \frac{1}{2}\xi - \frac{1}{2}\xi + \frac{1}{2}\xi^2 + \frac{1}{2}\xi^2 - \xi^2 + 1 = 1$$

(c) Calculate their derivatives respect to the natural coordinates.

$$\frac{\partial N_1}{\partial \xi} = \xi - \frac{1}{2} \qquad \frac{\partial N_2}{\partial \xi} = \xi + \frac{1}{2} \qquad \frac{\partial N_3}{\partial \xi} = 2\xi$$

Problem 5.2

A five node quadrilateral element has the nodal configuration shown if the figure with two perspective views of N_1^e and N_5^e. Find five shape functions $N_i^e, i = 1, ..., 5$ that satisfy compatibility and also verify that their sum is unity.

- *First of all, the shape function N_5 is obtained by construction as:*

$$N_5 = C_5\, L_{12}\, L_{23}\, L_{34}\, L_{41} \quad \Longrightarrow \quad C_5\,(1-\xi)(1-\eta)(1+\xi)(1+\eta)$$

- *Substituting the coordinates $(0, 0)$ corresponding with N_5 location; the value of C_5 is found as 1, arising the final shape-function:*

$$N_5 = (1-\xi^2)(1-\eta^2)$$

- *The given shape-functions for the corner nodes are the following:*

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta) \qquad N_2 = \frac{1}{4}(1+\xi)(1-\eta) \qquad N_3 = \frac{1}{4}(1+\xi)(1+\eta) \qquad N_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

- It is obtained a general shape function for all the nodes of the element:

$$N_i = \frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i) + \alpha N_5$$

- The value of α can be found substituting the coordinates of node 5 $(0,0)$ into any of the shape functions and equalize to zero. For instance, substitution of $\xi = \eta = 0$ in N_1 yields:

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta) + \alpha(1-\xi^2)(1-\eta^2) \quad (1)$$

$$= \frac{1}{4} + \alpha = 0 \qquad \Longrightarrow \qquad \alpha = -\frac{1}{4} \quad (2)$$

- In addition, the summation of all shape functions is equal to 1:

$$N_1+N_2+N_3+N_4+N_5 = \frac{1}{4}(1-\xi)(1-\eta)+\frac{1}{4}(1+\xi)(1-\eta)+\frac{1}{4}(1+\xi)(1+\eta)+\frac{1}{4}(1-\xi)(1+\eta)+4\alpha N_5+N_5$$

$$\Longrightarrow \frac{1}{4}(1-\xi-\eta+\xi\eta)+\frac{1}{4}(1+\xi-\eta-\xi\eta)+\frac{1}{4}(1+\xi+\eta+\xi\eta)+\frac{1}{4}(1-\xi+\eta-\xi\eta) - N_5 + N_5 = 1$$

Problem 5.3

Which minimum integration rules of Gauss-product type gives a rank sufficient stiffness matrix for these elements:
Before start computing the gauss points we are going to define the following values, taking account we are in 3D space.

$$n_F = 3n \quad number\ of\ element\ DoF \quad (3)$$
$$n_R = 6 \quad number\ of\ independent\ rigid\ body\ modes \quad (4)$$
$$n_E = 6 \quad order\ of\ E\ stress-strain\ matrix \quad (5)$$
$$n_G = \quad number\ of\ Gauss\ points\ in\ integration\ rule\ for\ K \quad (6)$$
$$r = \quad actual\ rank\ stiffness\ matrix \quad (7)$$

1. the 8-node hexahedron

$$r_{min} = 18 \qquad d = 0 \qquad with\ \mathbf{3}\ Gauss\ points$$

2. the 20-node hexahedron

$$r_{min} = 54 \qquad d = 0 \qquad with\ \mathbf{9}\ Gauss\ points$$

3. the 27-node hexahedron

$$r_{min} = 75 \qquad d = 0 \qquad with\ \mathbf{13}\ Gauss\ points$$

4. the 64-node hexahedron

$$r_{min} = 186 \qquad d = 0 \qquad with\ \mathbf{31}\ Gauss\ points$$