Universitat Politècnica de Catalunya
Master of Science in Computational Mechanics
Computational Structural Mechanics and Dynamics
CSMD
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# Assignment 5 - Isoparametric Representation and Convergence Requirements

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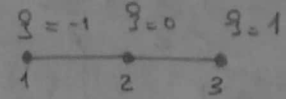
5.1

a) Given:




$N_1(\xi) = a_0 + a_1 \xi + a_2 \xi^2$ ; $N_2(\xi) = b_0 + b_1 \xi + b_2 \xi^2$

$N_3(\xi) = c_0 + c_1 \xi + c_2 \xi^2$

for $N_1$, the compatibility rezos:

$N_1(-1) = 1 = a_0 - a_1 + a_2$

$N_1(0) = 0 = a_0$

$N_1(1) = 0 = a_0 + a_1 + a_2$

$a_0 = 0$ ; $1 = 2a_2 \Rightarrow a_2 = \frac{1}{2} \Rightarrow a_1 = \frac{-1}{2}$

thus:

$$N_1(\xi) = \frac{1}{2}(\xi - 1)\xi$$

for $N_2$, the compatibility rezos:

$N_2(-1) = 0 = b_0 - b_1 + b_2$

$N_2(0) = 1 = b_0$

$N_2(1) = 0 = b_0 + b_1 + b_2$

$b_0 = 1$ ; $-2 = 2b_2 \Rightarrow b_2 = -1$ ; $b_1 = 0$

thus:

$$N_2(\xi) = 1 - \xi^2$$

for $N_3$, the compatibility rezos:

$N_3(-1) = 0 = c_0 - c_1 + c_2$ ; $c_0 = 0$ ; $2c_2 = 1 \Rightarrow c_2 = \frac{1}{2} \Rightarrow c_1 = \frac{1}{2}$

$N_3(0) = 0 = c_0$

$N_3(1) = 1 = c_0 + c_1 + c_2$

thus:

$$N_3(\xi) = \frac{1}{2}(\xi + 1)\xi$$

b)

$$N_1(\xi)+N_2(\xi)+N_3(\xi) = \frac{1}{2}(\xi-1)\xi + (1-\xi^2) + \frac{1}{2}(\xi+1)\xi$$

$$= \frac{1}{2}\xi^2 - \frac{\xi}{2} + 1 - \xi^2 + \frac{\xi}{2} + \frac{1}{2}\xi^2$$

$$= \frac{1}{2}\xi^2 + \frac{1}{2}\xi^2 - \xi^2 + 1$$

$$N_1(\xi)+N_2(\xi)+N_3(\xi) = 1$$

c)

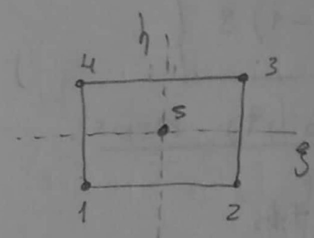
$$\frac{dN_1}{d\xi} = \frac{d}{d\xi}\left(\frac{1}{2}\xi^2 - \frac{1}{2}\xi\right) = \xi - \frac{1}{2}$$

$$\frac{dN_2}{d\xi} = \frac{d}{d\xi}\left(1-\xi^2\right) = -2\xi$$

$$\frac{dN_3}{d\xi} = \frac{d}{d\xi}\left(\frac{\xi^2}{2} + \frac{\xi}{2}\right) = \xi + \frac{1}{2}$$

5.2

by the Line method we Propose:




$$N_5(\xi,\eta) = C_1 L_1 L_2 \dots L_m$$

$$N_5(\xi,\eta) = C_1 L_{1-2} L_{2-3} L_{3-4} L_{4-1}$$

where:

along $L_{1-2}$ : $\eta = -1$ => $L_{1-2} = \eta + 1$

along $L_{2-3}$ : $\xi = 1$ => $L_{1-3} = \xi - 1$

along $L_{4-3}$ : $\eta = 1$ => $L_{4-3} = \eta - 1$

along $L_{4-1}$ : $\xi = -1$ => $L_{4-1} = \xi + 1$

thus:

$$N_5 = C_1(1+\xi)(1-\xi)(1+\eta)(1-\eta)$$

for compatibility

$N_5(0,0) = 1 = C_1 \quad \Rightarrow \quad C_1 = 1$

$N_5 = (1+\xi)(1-\xi)(1+\eta)(1-\eta)$

$$\boxed{N_5 = (1-\xi^2)(1-\eta^2)}$$

for the 4 corner nodes we have:

$\bar{N}_1 = C_1 L_{2-3} L_{4-3} = C_1(\xi-1)(\eta-1)$

$\bar{N}_1(-1,-1) = 1 = 4C_1 \quad \Rightarrow \quad C_1 = \frac{1}{4}$

$$\boxed{\bar{N}_1 = \frac{1}{4}(1-\xi)(1-\eta)}$$

$\bar{N}_2 = C_2 L_{3-4} L_{4-1} = C_2(\xi+1)(\eta-1)$

$\bar{N}_2(1,-1) = 1 = -4C_2 \quad \Rightarrow \quad C_2 = -\frac{1}{4}$

$$\boxed{\bar{N}_2 = \frac{1}{4}(1+\xi)(1-\eta)}$$

$\bar{N}_3 = C_3 L_{4-1} L_{1-2} = C_3(1+\xi)(1+\eta)$

$\bar{N}_3(1,1) = 1 = 4C_3 \quad \Rightarrow \quad C_3 = \frac{1}{4}$

$$\boxed{\bar{N}_3 = \frac{1}{4}(1+\xi)(1+\eta)}$$

$\bar{N}_4 = C_4 L_{1-2} L_{2-3} = C_4(\xi-1)(1+\eta)$

$\bar{N}_4(-1,1) = 1 = -4C_4 \Rightarrow \quad C_4 = -\frac{1}{4}$

$$\boxed{\bar{N}_4 = \frac{1}{4}(1-\xi)(1+\eta)}$$

for full compatibility $N_1, N_2, N_3$ and $N_4$ must vanish at node 5:
using the hierarchical correction technique we can write:

$N_i = \bar{N}_i + \alpha N_5 = \frac{1}{4}(1-\xi)(1-\eta) + \alpha(1-\xi^2)(1-\eta^2)$

$N_1(0,0) = 0 = \frac{1}{4} + \alpha \quad \Rightarrow \quad \alpha = -\frac{1}{4}$

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same $\alpha$ value is found for $N_2, N_3$ and $N_4$

thus:

$$N_1 = \frac{1}{4}(1-s)(1-h) - \frac{1}{4}(1-s^2)(1-h^2)$$

$$= \frac{1}{4}(1-s)(1-h) - \frac{1}{4}(1-s)(1-h)(1+s)(1+h)$$

$$\boxed{N_1 = \frac{1}{4}(1-s)(1-h)\left[1-(1+s)(1+h)\right]}$$

$$N_2 = \frac{1}{4}(1+s)(1-h) - \frac{1}{4}(1+s)(1-h)(1-s)(1+h)$$

$$\boxed{N_2 = \frac{1}{4}(1+s)(1-h)\left[1-(1-s)(1+h)\right]}$$

$$N_3 = \frac{1}{4}(1+s)(1+h) - \frac{1}{4}(1+s)(1+s)(1-s)(1-h)$$

$$\boxed{N_3 = \frac{1}{4}(1+s)(1+h)\left[1-(1-s)(1-h)\right]}$$

$$N_4 = \frac{1}{4}(1-s)(1+h) - \frac{1}{4}(1-s)(1+h)(1+s)(1-h)$$

$$\boxed{N_4 = \frac{1}{4}(1-s)(1+h)\left[1-(1+s)(1-h)\right]}$$

$$\boxed{N_5 = (1-s^2)(1-h^2)}$$

- Interpolation condition is now satisfied by all $N_i(s,h)$
- Local suport condition is also fullfilled by $N_1, N_2, N_3$ and $N_4$ as they vanish in boundaries where $i$ node doesn't appear
- Interelement compatibility condition is proved also, as follows:

for $N_1$: at $L_{1-2}$ $h = cte = -1 \Rightarrow N_1 = \frac{1}{2}(1-s)$ (Linear)

at $L_{4-1}$ $s = cte = -1 \Rightarrow N_1 = \frac{1}{2}(1-h)$ (Linear) for $N_2$: at $L_{1-2}$ $\eta = -1$ => $N_2 = \frac{1}{2}(1+\xi)$ (Linear)

at $L_{2-3}$ $\xi = 1$ => $N_2 = \frac{1}{2}(1-\eta)$ (Linear)

for $N_3$: at $L_{2-3}$ $\xi = 1$ => $N_3 = \frac{1}{2}(1+\eta)$ (Linear)

at $L_{4-3}$ $\eta = 1$ => $N_3 = \frac{1}{2}(1+\xi)$ (Linear)

for $N_4$: at $L_{4-3}$ $\eta = 1$ => $N_4 = \frac{1}{2}(1-\xi)$ (Linear)

at $L_{4-1}$ $\xi = -1$ => $N_4 = \frac{1}{2}(1+\eta)$ (Linear)

- thus, at boundaries all $N_i(\xi,\eta)$ are Linear functions ($1^{st}$ order) which requires 2 nodes:

- the summation of the shape functions is given by:

$$N_1 + N_2 + N_3 + N_4 + N_5 = \underbrace{\frac{1}{4}(1-\xi)(1-\eta) - \frac{1}{4}(1-\xi^2)(1-\eta^2)}_{N_1} +$$

$$+ \underbrace{\frac{1}{4}(1+\xi)(1-\eta) - \frac{1}{4}(1-\xi^2)(1-\eta^2)}_{N_2} + \underbrace{\frac{1}{4}(1+\xi)(1+\eta) - \frac{1}{4}(1-\xi^2)(1-\eta^2)}_{N_3} +$$

$$+ \underbrace{\frac{1}{4}(1-\xi)(1+\eta) - \frac{1}{4}(1-\xi^2)(1-\eta^2)}_{N_4} + (1-\xi^2)(1-\eta^2)$$

$$N_1 + N_2 + N_3 + N_4 + N_5 = \frac{1}{4}\left[(1-\xi)(1-\eta) + (1+\xi)(1-\eta) + (1+\xi)(1+\eta) + (1-\xi)(1+\eta)\right] +$$

$$+ (1-\xi^2)(1-\eta^2)\left[1 - \frac{1}{4} - \frac{1}{4} - \frac{1}{4} - \frac{1}{4}\right] \quad (= 0)$$

$$= \frac{1}{4}\left\{(1-\xi)\left[(1-\eta) + (1+\eta)\right] + (1+\xi)\left[(1-\eta)(1+\eta)\right]\right]$$

$$= \frac{1}{4}\left[2(1-\xi) + 2(1+\xi)\right] = \frac{1}{4}(2+2) = 1$$

thus:

$$\sum_{i=1}^{5} N_i = 1$$

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5.3

Given that, for a rank sufficiency elemental stiffness matrix we have:

$$r = m_F - m_R$$

where $m_f$ is the number of degrees of freedom of the element, and $m_R$ the number of independent rigid body modes.

as each Gauss point adds $m_E$ to the rank of $\underline{\underline{K}}^e$, to attain rank sufficiency, a numerical integration by Gauss quadrature must use a number of Gauss points $m_G$ such that

$$m_E m_G \geq m_f - m_R$$

here we define $m_E$ as the rank of the elasticity matrix denoted as $\underline{\underline{E}}$.

1) <u>8-node hexahedron</u>

- 3D element
- $m_E = 6$ ($\underline{\underline{E}}$ is a 6x6 matrix)
- $m_f = 3\times 8 = 24$ (3 degree of freedom per node)
- $m_R = 6$ (3 rotations and 3 translations)

thus:

$$r = m_f - m_R = 24 - 6 = 18$$

$$6 m_G \geq 18 \implies m_G \geq 3$$

hence: a Gauss rule <u>2x2x2</u> provides rank sufficient matrix as 1x1x1 provides only 1 Gauss point

2) <u>20-node hexahedron</u>

$m_E = 6$

$m_R = 6$

$m_f = 3\times 20 = 60$

thus:

$$r = m_f - m_R = 60 - 6 = 54$$

$$6 m_G \geq 54$$

$$m_G \geq 9$$

hence: a Gauss rule 2x2x2 gives only 8 points, then, a Gauss rule <u>3x3x3</u> is necessary

3) 27-node hexahedron

$m_E = 6$

$m_R = 6$

$m_f = 3 \times 27 = 81$

thus:

$r = m_f - m_R = 81 - 6 = 75$

$6 m_G \geqslant 75$

$m_G \geqslant 12{,}5$

hence, again a Gauss rule 3x3x3 is needed

4) 64-node hexahedron

$m_E = 6$

$m_R = 6$

$m_f = 3 \times 64 = 192$

thus:

$r = m_f - m_R = 192 - 6 = 186$

$6 m_G \geqslant 186$

$m_G \geqslant 31$

hence, a Gauss quadrature rule 3x3x3 only provides $m_G = 27$, thus a Gauss rule 4x4x4 is needed