Master on Numerical Methods in Engineering

Computational Structural Mechanics and Dynamics

Assignment 5

Convergence requirements

Mónica Ortega Castro

ASSIGNMENT 5. CONVERGENCE REQUIREMENTS

MÓNICA ORTEGA CASTRO

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Assignment 5.1: 1D.

Isoparametric definition of the three-node bar in its local system $\bar{x}$ is:

$$\begin{bmatrix} 1 \\ \bar{x} \\ \bar{u} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ \bar{x}_1 & \bar{x}_2 & \bar{x}_3 \\ \bar{u}_1 & \bar{u}_2 & \bar{u}_3 \end{bmatrix} \begin{bmatrix} N_1^e(\xi) \\ N_2^e(\xi) \\ N_3^e(\xi) \end{bmatrix} \qquad [1]$$

where:

- ξ is the isoparametric coordinate that takes values $-1, 1, 0$ at nodes $1, 2, 3$.
- N_1^e, N_2^e, N_3^e are the shape function of a bar element.

For simplicity, take values:

$$\bar{x}_1 = 0$$
$$\bar{x}_2 = L$$
$$\bar{x}_3 = \frac{1}{2}\ell + \alpha\ell = \ell(1/2 + \alpha)$$

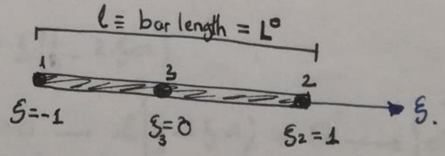
where:

- ℓ is the bar length.
- α characterises how far node 3 is away from the midpoint location $\bar{x} = \frac{1}{2}\ell$.

Show that the minimum α (minimal in absolute value sense) for which $J = d\bar{x}/d\xi$ vanishes at a point in the element are $\pm 1/4$ (the quarter points). Interpret this result as a singularity by showing that the axial strain (B) becomes infinite at an end point.

- Bar element to analyze:

  Three-noded bar in local system.



- Shape function definition:

  Given bar is a quadratic linear bar element, which means that the shape function must be parabolic.

  In order to obtain the SF for n nodes, it will be used: $N_j = \prod_{i=1,\, i\neq j,k}^{n} \frac{(\xi - \xi_i)(\xi - \xi_k)}{(\xi_i - \xi_j)(\xi_i - \xi_k)}$

  SFs also must satisfy $1 = \sum N_i$, according to [1], first equation.

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Shape function for each node:

$$N_1^e = \frac{\xi}{2}(\xi-1)$$

$i=1$
$j=2$
$k=3$

$$N_2^e = \frac{\xi}{2}(\xi+1)$$

$i=2$
$j=3$
$k=1$

$$N_3^e = 1 - N_2^e - N_1^e = 1-\xi^2$$

$i=3$
$j=1$
$k=2$

- Geometry interpolation:

$$x = \sum_{i=1}^{3} x_i N_i \quad \text{given in [1], second equation.}$$

x_i are the cartesian coordinates given as data.

$$x = \bar{x}_1 \cdot N_1^e + \bar{x}_2 \cdot N_2^e + \bar{x}_3 N_3^e = 0 + L \cdot \frac{\xi}{2} \cdot (\xi+1) + l\left(\frac{1}{2}+\alpha\right)(1-\xi^2).$$

- Obtaining the Jacobian, which in this case is a scalar: In this case $l=L$

$$J = \frac{dx}{d\xi} = \frac{l}{2}(2\xi+1) - 2\xi \cdot l\left(\frac{1}{2}+\alpha\right) = l\left(\xi + \frac{1}{2} - \xi - 2\xi\alpha\right) = l\left(\frac{1}{2} - 2\xi\alpha\right)$$

Jacobian vanishes at $J=0$; looking for α value (minimal in absolute value sense):

$$J = l\left(\frac{1}{2} - 2\cdot\xi\alpha\right)$$

$$\text{if } J=0 \longrightarrow \cancel{l}\left(\frac{1}{2} - 2\xi\cdot\alpha\right) = 0 \longrightarrow \boxed{\alpha = \pm 1/4}$$

The strain-displacement matrix B is defined by:

$$e = Bu^e = \frac{dN}{dx} \cdot u^e$$

if using it in the isoparametric representation: $B = \frac{dN}{d\xi} \cdot \frac{d\xi}{dx} = \frac{dN}{d\xi} \cdot J^{-1}$

if $J=0 \longrightarrow \alpha = \pm 1/4$.

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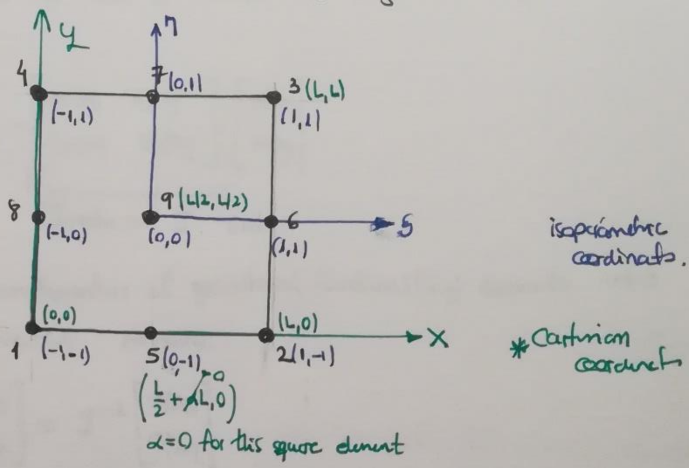
Assignment (5.2) : 2D.

Extend the results obtained from the previous exercise for a 9-node plane stress element. The element is initially a perfect square, nodes 5, 6, 7, 8 are at the midpoint of the sides 1-2, 2-3, 3-4, 4-1 and 9 at the center of the square. Move node 5 tangentially towards 2 until the Jacobian determinant at 2 vanishes. This result is important in the construction of "singular elements" for fracture mechanics.

- PLANE STRESS ELEMENT (2D)

  9-noded quadrilateral element (according to the description).

  Biquadratic behaviour.



- Shape functions:

→ At the corners:

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta)\xi\eta$$

$$N_2 = -\frac{1}{4}(1+\xi)(1-\eta)\xi\eta$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta)\xi\eta$$

$$N_4 = -\frac{1}{4}(1-\xi)(1+\eta)\xi\eta$$

→ In the mid-points:

$$N_5 = -\frac{1}{2}(1-\xi^2)(1-\eta)\eta$$

$$N_6 = \frac{1}{2}(1+\xi)(1-\eta^2)\xi$$

$$N_7 = \frac{1}{2}(1-\xi^2)(1+\eta)\cdot\eta$$

$$N_8 = -\frac{1}{2}(1-\xi)(1-\eta^2)$$

→ At the central point:

$$N_9 = (1-\xi^2)(1-\eta^2)$$

The isoparametric definition of the 9-noded biquadratic quadrilateral element is:

$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_9 \\ y_1 & y_2 & y_3 & \cdots & y_9 \\ u_{x1} & u_{x2} & u_{x3} & \cdots & u_{x9} \\ u_{y1} & u_{y2} & u_{y3} & \cdots & u_{y9} \end{bmatrix} \cdot \begin{bmatrix} N_1^e \\ N_2^e \\ \vdots \\ N_9^e \end{bmatrix} \quad [1]$$

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- Geometry interpolation: for this plane stress element (2D) in isoparametric coordinates.

Taking 2nd and 3rd equations from [1]:

$$x(\xi,\eta) = \sum_{i=1}^{9} x_i N_i(\xi,\eta).$$
$$y(\xi,\eta) = \sum_{i=1}^{9} y_i N_i(\xi,\eta).$$

Applying the chain rule in order to compute the derivative respect to the Cartesian coordinates:

$$\begin{bmatrix} \partial/\partial\xi \\ \partial/\partial\eta \end{bmatrix} = \underbrace{\begin{bmatrix} \partial x/\partial\xi & \partial y/\partial\xi \\ \partial x/\partial\eta & \partial y/\partial\eta \end{bmatrix}}_{\text{Jacobian } J \;\; 2\text{x}2} \cdot \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \end{bmatrix}$$

So that, the transformation of geometrical functions (x,y) coordinates into cartesian coordinates remains:

$$\begin{bmatrix} \partial/\partial x \\ \partial/\partial y \end{bmatrix} = J^{-1} \begin{bmatrix} \partial/\partial\xi \\ \partial/\partial\eta \end{bmatrix}$$

- In order to describe the movement of node 5 tangentially to node 2 until the Jacobian determinant vanishes $(J=0)$ we first need to compute J: for the 9 nodes:

$$J = \begin{bmatrix} \partial N_i/\partial\xi \\ \partial N_i/\partial\eta \end{bmatrix} [x_i \;\; y_i] = \begin{bmatrix} \sum_{i=1}^{9} x_i \partial N_i/\partial\xi & \sum_{i=1}^{9} y_i \partial N_i/\partial\eta \\ \sum_{i=1}^{9} x_i \partial N_i/\partial\eta & \sum_{i=1}^{9} y_i \partial N_i/\partial\eta \end{bmatrix} \qquad [2]$$

* See the cartesian representation and assigned coordinates to each node in the element's graphical representation *.

For the mid nodes the portion along the axis is $\frac{L}{2} + \alpha L$ where $-\frac{1}{2} < \alpha < \frac{1}{2}$. Initially, as the statement said $\alpha = 0$ because the element is a perfect square.

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- Shape functions derivatives: [3]

→ w.r.t ξ:

$$\partial N_1/\partial \xi = \frac{\eta}{4}(1-\eta)(1-2\xi)$$
$$\partial N_2/\partial \xi = \frac{\eta}{4}(\eta-1)$$
$$\partial N_3/\partial \xi = \frac{\eta}{4}(1+\eta)(1+2\xi)$$
$$\partial N_4/\partial \xi = \frac{\eta}{4}(1+\eta)(2\xi-1)$$
$$\partial N_5/\partial \xi = \eta\xi(1-\eta)$$
$$\partial N_6/\partial \xi = \frac{1-\eta^2}{2}(1+2\xi)$$
$$\partial N_7/\partial \xi = -\eta\xi(1+\eta)$$
$$\partial N_8/\partial \xi = \frac{1-\eta^2}{2}(2\xi-1)$$
$$\partial N_9/\partial \xi = 2\xi(\eta^2-1)$$

→ w.r.t η:

$$\partial N_1/\partial \eta = \frac{\xi}{4}(1-\xi)(1-2\eta)$$
$$\partial N_2/\partial \eta = \frac{\xi}{4}(1+\xi)(2\eta-1)$$
$$\partial N_3/\partial \eta = \frac{\xi}{4}(1+\xi)(1+2\eta)$$
$$\partial N_4/\partial \eta = \frac{\xi}{4}(1-\xi)(-1-2\eta)$$
$$\partial N_5/\partial \eta = \frac{1-\xi^2}{2}(2\eta-1)$$
$$\partial N_6/\partial \eta = -\xi\eta(1+\xi)$$
$$\partial N_7/\partial \eta = \frac{1-\xi^2}{2}(1+2\eta)$$
$$\partial N_8/\partial \eta = \eta\xi(1-\xi)$$
$$\partial N_9/\partial \eta = 2\eta(\xi^2-1)$$

Working with [2], the Jacobian determinant is computed with the above calculations

$$J(\xi,\eta) = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

Replacing values for the node 2 whose isoparametric coordinates are $(1,-1)$:

$$J_{11} = L\left(\frac{1}{2}-2\alpha\right)$$
$$J_{12} = 0$$
$$J_{21} = L\left(\frac{-3}{2}\right) - \frac{L}{2} + 2L = 0$$
$$J_{22} = L/2.$$

→ Jacobian for node $2(1,-1)$ is: $J = \begin{bmatrix} L\left(\frac{1}{2}-2\alpha\right) & 0 \\ 0 & L/2 \end{bmatrix}$

Resulting determinant is: $|J| = \frac{L^2}{2}\left(\frac{1}{2}-2\alpha\right)$

For $|J| = 0 \rightarrow \frac{L^2}{2}\left(\frac{1}{2}-2\alpha\right) = 0 \rightarrow \alpha = 1/4$

While moving node 5 towards node 2, the Jacobian determinant vanishes when $\boxed{\alpha=1/4}$ $[J(1,-1)]$.

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