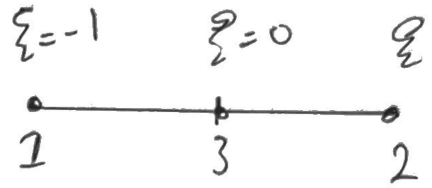
Assignment 5.1

$$\begin{bmatrix} 1 \\ \bar{x} \\ \bar{y} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ \bar{x}_1 & \bar{x}_2 & \bar{x}_3 \\ \bar{u}_1 & \bar{u}_2 & \bar{u}_3 \end{bmatrix} \begin{bmatrix} N_1^e(\xi) \\ N_2^e(\xi) \\ N_3^e(\xi) \end{bmatrix}$$

$\bar{x}_1 = 0$
$\bar{x}_2 = L$
$\bar{x}_3 = L/2 + \alpha L$



$N_1 = \frac{1}{2}\xi(\xi - 1)$

$N_2 = \frac{1}{2}\xi(\xi + 1)$

$N_3 = 1 - \xi^2$

$$\bar{x} = \sum_{i=1}^{3} N_i^e \bar{x}_e = \frac{L}{2}\xi(\xi+1) + \left(\frac{L}{2} + \alpha L\right)(1 - \xi^2)$$

$$= L\left[\xi^2/2 + \xi/2 + 1/2 - \xi^2/2 + \alpha - \alpha\xi^2\right]$$

$$= L\left[\xi/2 + 1/2 + \alpha - \alpha\xi^2\right]$$

$$J = \frac{d\bar{x}}{d\xi} = L\left[1/2 - 2\alpha\xi\right]$$

For J to vanish we need:

$$J = 0 \Rightarrow 1/2 = 2\alpha\xi$$

$$\Rightarrow \alpha = 1/4 \cdot \frac{1}{\xi}$$

Since : $-1 \leq \xi \leq 1$ we get that J vanishes for $\xi = 1$ and $\xi = -1$, at the end nodes.

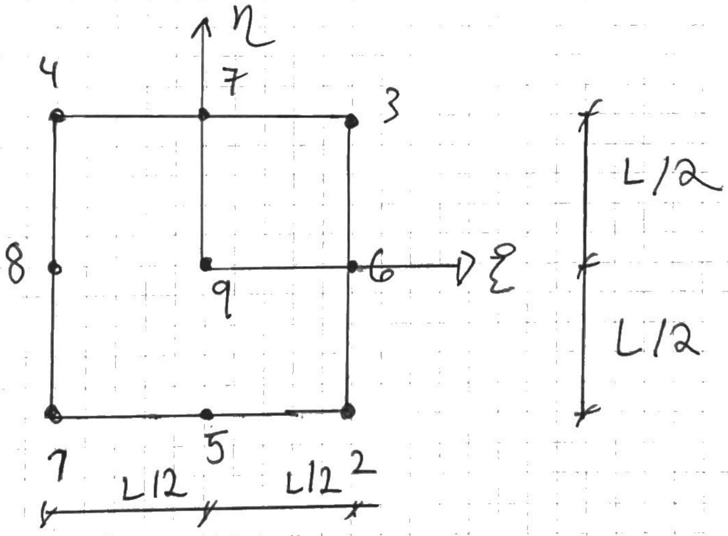
The axial strain is given by:

$$\varepsilon_x = \frac{du}{dx} = \frac{1}{J}\frac{du}{\partial \xi}$$, at the end

<u>points $J \to 0$ which means $\varepsilon_x \to \infty$.</u>

<u>Assignment 5.2</u>

Now:



Defining the shape functions:

$$N_1(\xi, \eta) = \frac{1}{4}\eta\xi(1-\xi)(1-\eta)$$

$$N_2(\xi, \eta) = -\frac{1}{4}\xi\eta(1+\xi)(1-\eta)$$

$$N_3(\xi, \eta) = \frac{1}{4}\xi\eta(1+\xi)(1+\eta)$$

$$N_4(\xi,\eta) = -\frac{1}{4}\xi\eta(1-\xi)(1+\eta)$$

$$N_5(\xi,\eta) = -\frac{1}{2}\eta(1-\xi^2)(1-\eta)$$

$$N_6(\xi,\eta) = \frac{1}{2}\xi(1+\xi)(1-\eta^2)$$

$$N_7(\xi,\eta) = \frac{1}{2}\eta(1-\xi^2)(1+\eta)$$

$$N_8(\xi,\eta) = -\frac{1}{2}\xi\eta(1-\xi)(1-\eta)$$

$$N_9(\xi,\eta) = (1-\xi^2)(1-\eta^2)$$

Since
$$J = \begin{bmatrix} \sum_{i=1}^{9} x_i \frac{\partial N_i}{\partial \xi} & \sum_{i=1}^{9} y_i \frac{\partial N_i}{\partial \xi} \\ \sum_{i=1}^{9} x_i \frac{\partial N_i}{\partial \eta} & \sum_{i=1}^{9} y_i \frac{\partial N_i}{\partial \eta} \end{bmatrix}$$

Derivatives:

$$\left.\begin{aligned} \frac{\partial N_1}{\partial \xi} &= \frac{1}{4}\eta(1-\eta)(1-2\xi) \\ \frac{\partial N_1}{\partial \eta} &= \frac{1}{4}\xi(1-\xi)(1-2\eta) \end{aligned}\right\} N_1$$

$$\left.\begin{aligned} \frac{\partial N_2}{\partial \xi} &= -\frac{1}{4}\eta(1-\eta)(1+2\xi) \\ \frac{\partial N_2}{\partial \eta} &= -\frac{1}{4}\xi(1+\xi)(1-2\eta) \end{aligned}\right\} N_2$$

$$\left.\begin{aligned}
\frac{\partial N_3}{\partial \xi} &= \frac{1}{4}\eta(1+\eta)(1+2\xi) \\
\frac{\partial N_3}{\partial \eta} &= \frac{1}{4}\xi(1+\xi)(1+2\eta)
\end{aligned}\right\} N_3$$

$$\left.\begin{aligned}
\frac{\partial N_4}{\partial \xi} &= \frac{-1}{4}\eta(1+\eta)(1-2\xi) \\
\frac{\partial N_4}{\partial \eta} &= \frac{-1}{4}\xi(1-\xi)(1+2\eta)
\end{aligned}\right\} N_4$$

$$\left.\begin{aligned}
\frac{\partial N_5}{\partial \xi} &= \frac{-1}{2}\eta(1-\eta)(-2\xi) \\
\frac{\partial N_5}{\partial \eta} &= \frac{-1}{2}(1-\xi^2)(1-2\eta)
\end{aligned}\right\} N_5$$

$$\left.\begin{aligned}
\frac{\partial N_6}{\partial \xi} &= \frac{1}{2}(1-\eta^2)(1+2\xi) \\
\frac{\partial N_6}{\partial \eta} &= \frac{1}{2}\xi(1+\xi)(-2\eta)
\end{aligned}\right\} N_6$$

$$\left.\begin{aligned}
\frac{\partial N_7}{\partial \xi} &= \frac{1}{2}\eta(1+\eta)(1-2\xi) \\
\frac{\partial N_7}{\partial \eta} &= \frac{1}{2}(1-\xi^2)(1+2\eta)
\end{aligned}\right\} N_7$$

$$\left.\begin{aligned} \frac{\partial N_8}{\partial \xi} &= \frac{-1}{2}\eta(1-\eta)(1-2\xi) \\[2ex] \frac{\partial N_8}{\partial \eta} &= \frac{-1}{2}\xi(1-\xi)(1-2\eta) \end{aligned}\right\} N_8$$

$$\left.\begin{aligned} \frac{\partial N_9}{\partial \xi} &= (1-2\xi)(1-\eta^2) \\[2ex] \frac{\partial N_9}{\partial \eta} &= (1-\xi^2(1-2\eta) \end{aligned}\right\} N_9$$

Now we are going to evaluate the Jacobian at node 2 $\Rightarrow$ $(\xi,\eta) = (1,-1)$

At $(\xi,\eta) = (-1,1)$ we have:

$N_{1,\xi} = 1/2$ $\qquad$ $N_{3,\xi} = 0$ $\qquad$ $N_{5,\xi} = -2$

$N_{1,\eta} = 0$ $\qquad$ $N_{3,\eta} = -1/2$ $\qquad$ $N_{5,\eta} = 0$

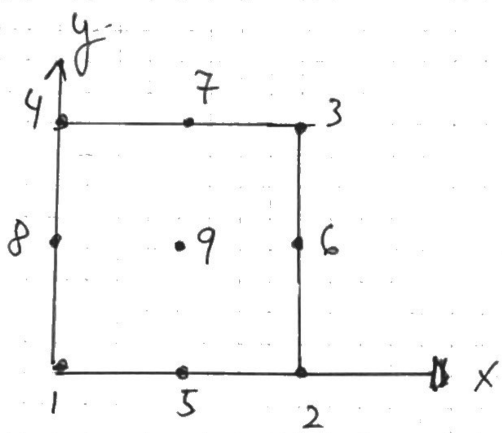
$N_{2,\xi} = 3/2$ $\qquad$ $N_{4,\xi} = 0$ $\qquad$ $N_{6,\xi} = 0$

$N_{2,\eta} = -3/2$ $\qquad$ $N_{4,\eta} = 0$ $\qquad$ $N_{6,\eta} = 2$

$N_{7,\xi} = 0$ $\qquad$ $N_{8,\xi} = 0$ $\qquad$ $N_{9,\xi} = 0$

$N_{7,\eta} = 0$ $\qquad$ $N_{8,\eta} = 0$ $\qquad$ $N_{9,\eta} = 0$

If we consider node 6, 7, 8 fixed at mid-point we can consider the following geometry:



Node:	Coordinates:
1	$(0,0)$
2	$(L,0)$
3	(L,L)
4	$(0,L)$
5	$(L/2+\alpha L, 0)$
6	$(L, L/2)$
7	$(L/2, L)$
8	$(0, L/2)$
9	$(L/2, L/2)$

$$\Rightarrow \sum_{i=1}^{9} x_i N_{i,\xi} = L \cdot N_{2,\xi} + L N_{3,\xi} + (L/2 + \alpha L) N_{5,\xi}$$
$$+ N_{6,\xi} \cdot L + N_{7,\xi} L/2 + N_{9,\xi} \cdot L/2$$

$$= 3/2\, L + 0 + (+L/2 + \alpha L) \cdot (-2) + 0 + 0$$
$$= L/2 - 2\alpha L$$

$$\sum_{i=1}^{9} x_i N_{i,\eta} = -3/2\, L - 1/2\, L + 2L = 0$$

$$\sum_{i=1}^{9} y_i N_{i,\xi} = L \cdot N_{3\xi} + L \cdot N_{4\xi} + L/2\, N_{6\xi} + L N_{7,\xi} + N_{8,\xi} \cdot L/2 + N_{9,\xi} \cdot L/2$$

$$= 0 + 0 + 0 + 0 + 0 + 0 = 0$$

$$\sum_{i=1}^{9} y_i N_{i,\eta} = -L/2 + 0 + \frac{2L}{2} + 0 + 0 + 0$$
$$= L/2$$

$$\Rightarrow \quad J(1,-1) = \begin{bmatrix} L/2 - 2\alpha L & 0 \\ 0 & L/2 \end{bmatrix}$$

$$\det J = L^2 (1/4 - \alpha)$$

$$\det J = 0 \quad \text{if} \quad \alpha = 1/4$$

This means that node 5 can be moved at max $L/4$ to the right, or else the Jacobian will be zero or negative in node 2.