Assignment 5. Computational Structural Mechanics and Dynamics

Exercise 5.1

3-node bar element

$$\begin{bmatrix} 1 \\ \tilde{x} \\ \bar{u} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ \tilde{x}_1 & \tilde{x}_2 & \tilde{x}_3 \\ \tilde{u}_1 & \tilde{u}_2 & \tilde{u}_3 \end{bmatrix} \begin{bmatrix} N_1^e(\xi) \\ N_2^e(\xi) \\ N_3^e(\xi) \end{bmatrix}$$

$$N_1^e(\xi) = \frac{1}{2}\xi(\xi-1)$$
$$N_3^e(\xi) = 1-\xi^2$$
$$N_2^e(\xi) = \frac{1}{2}\xi(\xi+1)$$

$$\tilde{x} = \sum N_i x_i = \tilde{x}_2 \cdot N_2 + \tilde{x}_3 \cdot N_3$$
$$= \frac{\xi L}{2}(\xi+1) + \left(\frac{L}{2} + \alpha L\right)(1-\xi^2)$$
$$= \frac{\xi L}{2}(\xi+1) + L\left(1/2 - \frac{\xi^2}{2} + \alpha - \alpha \cdot \xi^2\right)$$

$$J = \frac{d\tilde{x}}{d\xi} = \frac{L}{2}(\cancel{2}\xi+1) + L(-\cancel{\xi} - 2\alpha\xi) = \frac{L}{2}(1-4\alpha\xi)$$

when inserting $\alpha = 1/4$ then will $\xi = 1$ ← node 2 give $J=0$. When inserting $\alpha = -1/4$ then will $\xi = -1$ ← node 1. give $J=0$.

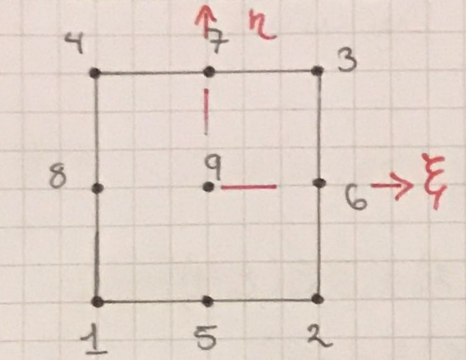
$$\tilde{u} = \tilde{u}_1 N_1^e(\xi) + \tilde{u}_2 N_2^e(\xi) + \tilde{u}_3 N_3^e(\xi)$$

$$\Rightarrow \varepsilon = \frac{d\tilde{u}}{dx} = \frac{d\tilde{u}}{d\xi} \cdot \underbrace{\frac{d\xi}{dx}}_{1/J} = \frac{1}{J}\left(\frac{dN_1}{d\xi}\tilde{u}_1 + \frac{dN_2}{d\xi}\tilde{u}_2 + \frac{dN_3}{d\xi}\tilde{u}_3\right)$$

$$= \frac{1}{J}\left(\frac{1}{2}(2\xi-1) \cdot \tilde{u}_1 + \frac{1}{2}(2\xi+1)\tilde{u}_2 - 2\xi\tilde{u}_3\right)$$

If $J \to 0$ then $\varepsilon \to \infty$ ⇒ Axial strain becomes infinite at an end points.

Exercise 5.2



$$N_1(\xi,\eta) = N_1(\xi)\,N_1(\eta) = \underline{\frac{\xi\eta}{4}(\xi-1)(\eta-1)}$$

$$N_2(\xi,\eta) = N_2(\xi)\,N_1(\eta) = \underline{\frac{\xi\eta}{4}(\xi+1)(\eta-1)}$$

$$N_3(\xi,\eta) = \underline{\frac{\xi\eta}{4}(\xi+1)(\eta+1)} \qquad N_4(\xi,\eta) = \underline{\frac{\xi\eta}{4}(\xi-1)(\eta+1)}$$

$$N_5(\xi,\eta) = \underline{(1-\xi^2)\frac{\eta}{2}(\eta-1)} \qquad N_6(\xi,\eta) = \underline{(1-\eta^2)\frac{\xi}{2}(\xi+1)}$$

$$N_7(\xi,\eta) = \underline{(1-\xi^2)\frac{\eta}{2}(\eta+1)} \qquad N_8(\xi,\eta) = \underline{(1-\eta^2)\frac{\xi}{2}(\xi-1)}$$

$$N_9(\xi,\eta) = \underline{(1-\xi^2)(1-\eta^2)}$$

$$x = x_1N_1 + x_2N_2 + x_3N_3 + x_4N_4 + x_5N_5 + x_6N_6 + x_7N_7 + x_8N_8 + x_9N_9$$

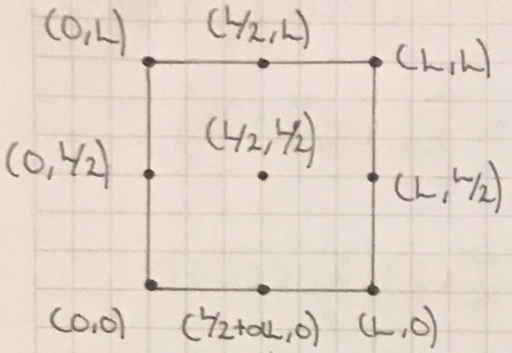
$$y = y_1N_1 + y_2N_2 + y_3N_3 + y_4N_4 + y_5N_5 + y_6N_6 + y_7N_7 + y_8N_8 + y_9N_9$$

$$J = \frac{\partial(x,y)}{\partial(\xi,\eta)} = \begin{bmatrix} \partial x/\partial\xi & \partial y/\partial\xi \\ \partial x/\partial\eta & \partial y/\partial\eta \end{bmatrix}$$

$$\frac{\partial x}{\partial\xi} = \sum x_i \frac{\partial N_i}{\partial\xi}, \quad \frac{\partial y}{\partial\xi} = \sum y_i \frac{\partial N_i}{\partial\xi}, \quad \frac{\partial x}{\partial\eta} = \sum x_i \frac{\partial N_i}{\partial\eta}$$

$$\frac{\partial y}{\partial\eta} = \sum y_i \cdot \frac{\partial N_i}{\partial\eta}$$

Move node $5 \rightarrow 2$



$$\Rightarrow x = LN_2 + LN_3 + (L/2 + \alpha L)N_5 + LN_6 + L/2\,N_7 + L/2\,N_9$$

$$y = LN_3 + LN_4 + \frac{L}{2}N_6 + LN_7 + L/2\,N_8 + L/2\,N_9$$

$$\frac{\partial x}{\partial \xi} = \frac{\eta L}{4}(2\xi+1)(\eta-1) + \frac{\eta L}{4}(2\xi+1)(\eta+1) + (L/2 + \alpha L)\cdot(-\xi\eta)(\eta-1)$$

$$+ \frac{L}{2}(2\xi+1)(1-\eta^2) + \frac{\eta L}{4}(-2\xi)(\eta+1) + \frac{L}{2}(-2\xi)(1-\eta^2)$$

at node 2.

evaluate at $(\xi,\eta) = (1,-1)$

$$\Rightarrow \left.\frac{\partial x}{\partial \xi}\right|_{(1,-1)} = \frac{6L}{4} + 0 + (L/2 + \alpha L)(-2) + 0 + 0 + 0$$

$$= \frac{6L}{4} - L - 2\alpha L = \underline{\frac{1}{2}L - 2\alpha L}$$

$$y = LN_3 + LN_4 + L/2\,N_6 + LN_7 + \frac{L}{2}N_8 + \frac{L}{2}N_9$$

$$\Rightarrow \frac{\partial y}{\partial \xi} = \frac{\eta L}{4}(2\xi+1)(\eta+1) + \frac{\eta L}{4}(2\xi-1)(\eta+1) + \frac{L}{4}(2\xi+1)(1-\eta^2)$$

$$+ \frac{\eta L}{2}(-2\xi)(\eta+1) + \frac{-L}{4}(2\xi+1)(1-\eta^2) + \frac{L}{2}(-2\xi)(1-\eta^2)$$

evaluate at $(\xi,\eta) = (1,-1)$

$$\left.\frac{\partial y}{\partial \xi}\right|_{(1,-1)} = 0 + 0 + 0 + 0 + 0 + 0 = \underline{0}$$

$$\frac{\partial x}{\partial \eta} = \frac{\xi_1 L}{4}(\xi_1+1)(2\eta-1) + \frac{\xi_1 L}{4}(\xi_1+1)(2\eta+1) + \left(\frac{L}{2}+\alpha L\right)\frac{1}{2}(1-\xi_1^2)(2\eta-1)$$

$$+ \frac{\xi_1 L}{2}(\xi_1+1)(-2\eta) + L/4(1-\xi_1^2)(2\eta+1) + \frac{L}{2}(1-\xi_1^2)(-2\eta)$$

$$\left.\frac{\partial x}{\partial \eta}\right|_{(1,-1)} = \frac{L}{4}(2)(-3) + \frac{L}{4}(2)(-1) + 0 + \frac{L}{2}(2)(2) + 0 + 0$$

$$= \frac{-6L}{4} - \frac{2L}{4} + 2L = \underline{0}$$

$$\frac{\partial y}{\partial \eta} = \frac{\xi_1 L}{4}(\xi_1+1)(2\eta+1) + \frac{\xi_1 L}{4}(\xi_1-1)(2\eta+1) + \frac{\xi_1 L}{4}(\xi_1+1)(-2\eta)$$

$$+ \frac{L}{2}(2\eta+1)(1-\xi_1^2) + \frac{L}{2}\cdot\frac{\xi_1}{2}(\xi_1-1)(-2\eta) + \frac{L}{2}(1-\xi_1^2)(-2\eta)$$

$$\left.\frac{\partial y}{\partial \eta}\right|_{(1,-1)} = \frac{L}{4}(2)(-1) + 0 + \frac{L}{4}(2)(2) + 0 + 0 + 0$$

$$= \frac{-L}{2} + L = \underline{\frac{L}{2}}$$

$$J^{(1,-1)} = \begin{bmatrix} L/2 - 2\alpha L & 0 \\ 0 & \frac{L}{2} \end{bmatrix} \Rightarrow \det(J)^{(1,-1)} = \underline{\frac{L}{2}\underbrace{(L/2 - 2\alpha L)}_{=0}}$$

$$\underline{\alpha = \frac{1}{4}}$$

Same as for the bar element in exercise 5.1