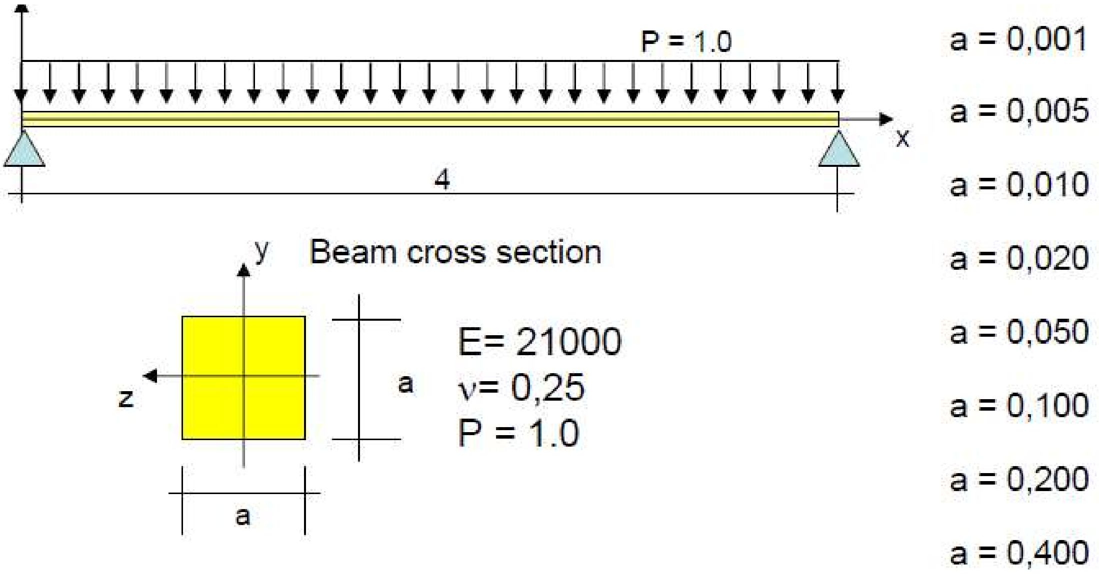
1) The shear stiffness matrix is

$$K_S = \begin{pmatrix} 1 & \frac{l}{2} & -1 & \frac{l}{2} \\ \frac{l}{2} & \frac{l^2}{4} & -\frac{l}{2} & \frac{l^2}{4} \\ -1 & -\frac{l}{2} & 1 & -\frac{l}{2} \\ \frac{l}{2} & \frac{l^2}{4} & -\frac{l}{2} & \frac{l^2}{4} \end{pmatrix}$$

To implement the Timoshenko 2 nodes beam element with reduce integration, the elemental shear stiffness matrix is coded as:

```
K_s = [ 1 , len/2 , -1 , len/2 ;
        len/2 , len^2/4 , -len/2 , len^2/4 ;
         -1 , -len/2 , 1 , -len/2 ; len/2 ,
         len^2/4 , -len/2 , len^2/4 ];
```

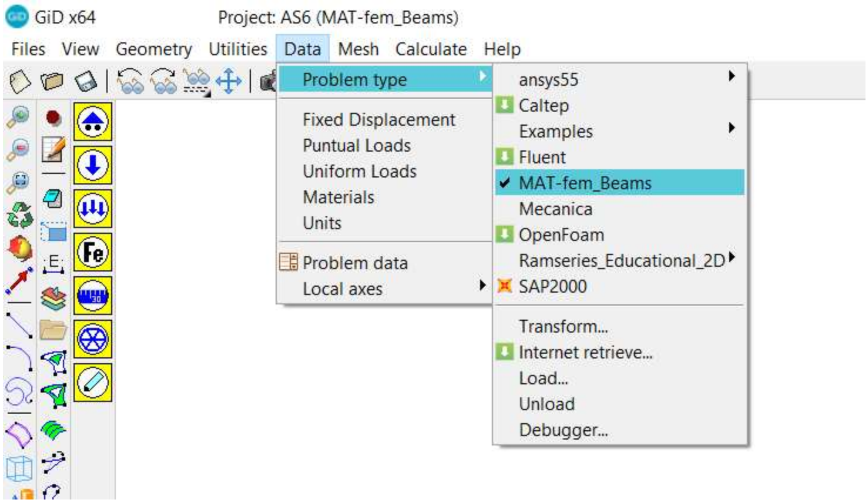
2) Solve the following problem with a 64 element mesh



Different configurations are considered for the beam cross section:

| a | Area | Inertia | L | a/L |
|---|---|---|---|---|
| 0,001 | 0,000001 | 8,33333E-14 | 4 | 0,00025 |
| 0,005 | 0,000025 | 5,20833E-11 | 4 | 0,00125 |
| 0,01 | 0,0001 | 8,33333E-10 | 4 | 0,0025 |
| 0,02 | 0,0004 | 1,33333E-08 | 4 | 0,005 |
| 0,05 | 0,0025 | 5,20833E-07 | 4 | 0,0125 |
| 0,1 | 0,01 | 8,33333E-06 | 4 | 0,025 |
| 0,2 | 0,04 | 0,000133333 | 4 | 0,05 |
| 0,4 | 0,16 | 0,002133333 | 4 | 0,1 |

After modifying the code for Timoshenko reduced integration, the results have been implemented by loading the problem type MAT-FEM Beams.



These are the results for the maximum deflection, maximum bending moment and maximum shear forces of the beam for different configurations of slenderness ratio a/L for 3 different 2-noded beam elements with a 64 element mesh.

a. 2 nodes Euler-Bernoulli element.

| Euler bernoulli | | | |
|---|---|---|---|
| a/L | Max displacement | Max Bending Moment | Max Shear |
| 0,00025 | 1,905E+09 | 2,000E+00 | 2,000E+00 |
| 0,00125 | 3,048E+08 | 2,000E+00 | 2,000E+00 |
| 0,0025 | 1,905E+05 | 2,000E+00 | 2,000E+00 |
| 0,005 | 1,191E+04 | 2,000E+00 | 2,000E+00 |
| 0,0125 | 3,048E+02 | 2,000E+00 | 2,000E+00 |
| 0,025 | 1,905E+01 | 2,000E+00 | 2,000E+00 |
| 0,05 | 1,191E+00 | 2,000E+00 | 2,000E+00 |
| 0,1 | 7,441E-02 | 2,000E+00 | 2,000E+00 |

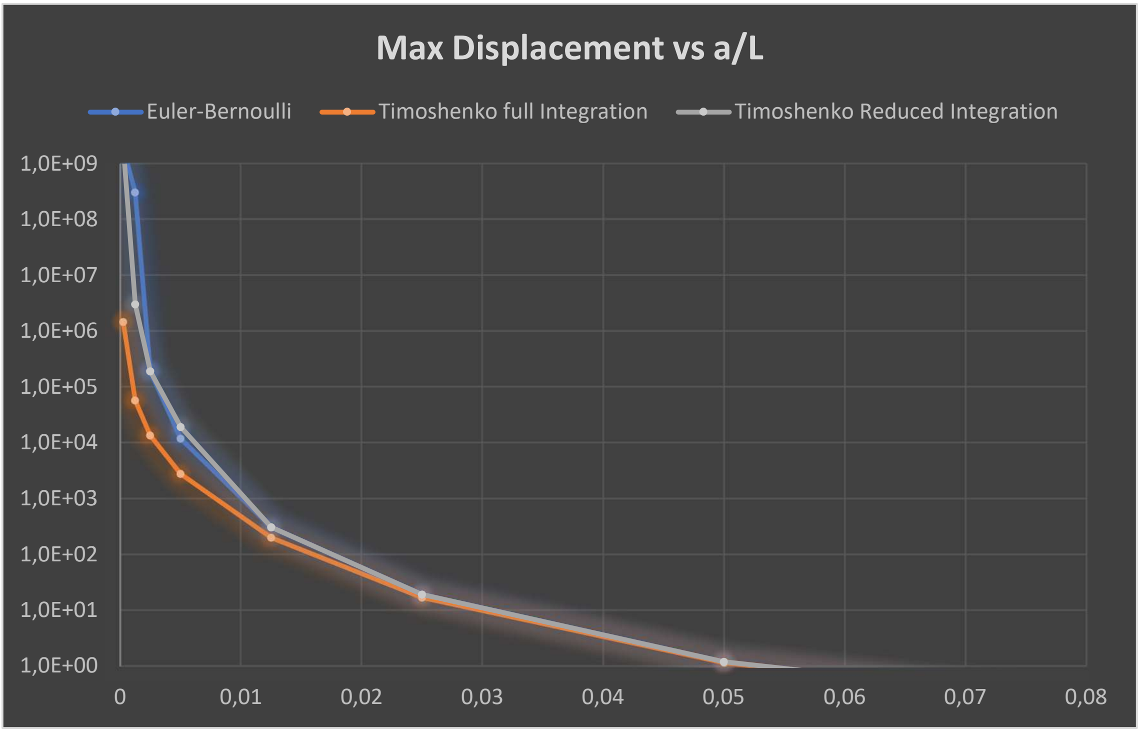
b. 2 nodes Timoshenko Full Integrate element

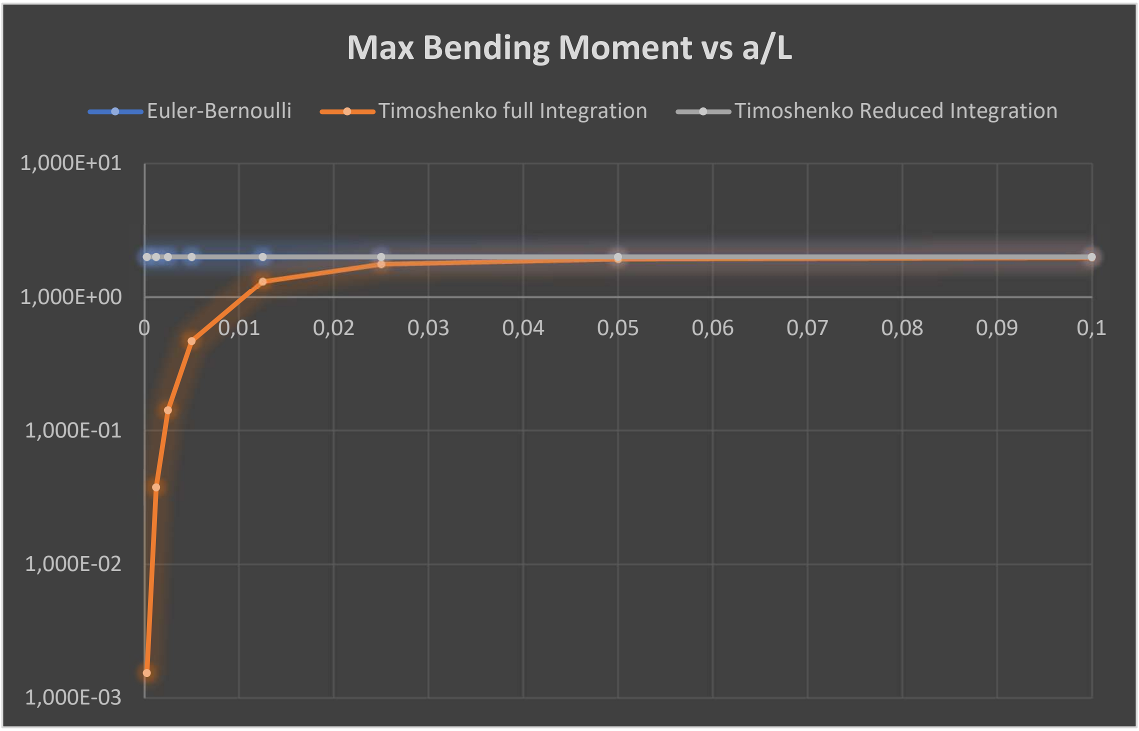
| Timoshenko full integration | | | |
|---|---|---|---|
| a/L | Max displacement | Max Bending Moment | Max Shear |
| 0,00025 | 1,462E+06 | 1,534E-03 | 1,969E+00 |
| 0,00125 | 5,740E+04 | 3,766E-02 | 1,969E+00 |
| 0,0025 | 1,358E+04 | 1,426E-01 | 1,969E+00 |
| 0,005 | 2,797E+03 | 4,698E-01 | 1,969E+00 |
| 0,0125 | 2,004E+02 | 1,314E+00 | 1,969E+00 |
| 0,025 | 1,688E+01 | 1,769E+00 | 1,969E+00 |
| 0,05 | 1,160E+00 | 1,936E+00 | 1,969E+00 |
| 0,1 | 7,556E-02 | 1,983E+00 | 1,969E+00 |

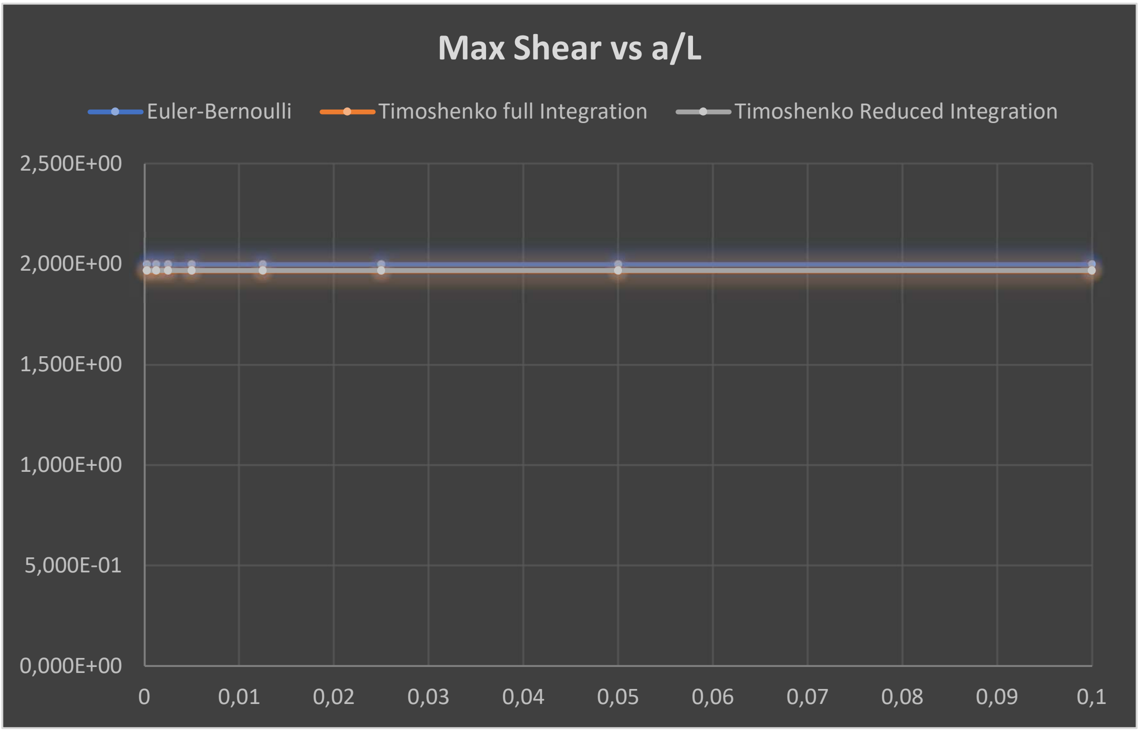
c. 2 nodes Timoshenko Reduce Integration element

| Timoshenko Reduced integration | | | |
|---|---|---|---|
| a/L | Max displacement | Max Bending Moment | Max Shear |
| 0,00025 | 1,904E+09 | 1,999E+00 | 1,969E+00 |
| 0,00125 | 3,046E+06 | 1,999E+00 | 1,969E+00 |
| 0,0025 | 1,904E+05 | 1,999E+00 | 1,969E+00 |
| 0,005 | 1,910E+04 | 1,999E+00 | 1,969E+00 |
| 0,0125 | 3,048E+02 | 1,999E+00 | 1,969E+00 |
| 0,025 | 1,907E+01 | 1,999E+00 | 1,969E+00 |
| 0,05 | 1,197E+00 | 1,999E+00 | 1,969E+00 |
| 0,1 | 7,616E-02 | 1,999E+00 | 1,969E+00 |

Here below there are the graphical results for all beams with respect to max. displacement, max. bending moment and max shear forces.




Max Bending Moment vs a/L
Euler-Bernoulli
Timoshenko full Integration
Timoshenko Reduced Integration
1,000E+01
1,000E+00
1,000E-01
1,000E-02
1,000E-03
0
0,01
0,02
0,03
0,04
0,05
0,06
0,07
0,08
0,09
0,1


Max Shear vs a/L
Euler-Bernoulli
Timoshenko full Integration
Timoshenko Reduced Integration
2,500E+00
2,000E+00
1,500E+00
1,000E+00
5,000E-01
0,000E+00
0
0,01
0,02
0,03
0,04
0,05
0,06
0,07
0,08
0,09
0,1

We can reach several conclusions about the different theories after analyzing the results:

Displacements

It can be noticed that as a/L increases, the three models converge to similar results in the displacements. when shear locking occurs, the element becomes more rigid, consequently the element suffers less displacement.
For low slenderness ratio values 'a/L'. we notice that the Euler Bernoulli and Reduced Timoshenko beam models show similar and good results. However, the Timoshenko full integration beam shows a lower displacement value.

Bending moment

Considering the analytical value of the maximum bending moment, it is seen that the Timoshenko full integration beam does not exhibit good results initially, due to shear locking. The moments obtained for this method start being similar to the analytical moment for a slenderness value of a/L > 0:05
It can be noticed that the Euler-Bernoulli and Timoshenko Reduce Integration models give very accurate results.
Whereas the Timoshenko reduced integration and full integration show almost exactly the same result. for all values of 'a/L'

Shear forces

For the Shear graphs, the three methods show similar results which work good for all values of the ratio.

As a conclusion, Euler-Bernoulli model is very accurate for the bending moment calculation in a slender beam problem. Timoshenko's beam model performs better in a shear case-problem, however, full integrate Timoshenko could not work properly since it brings distortion in calculations (increment of stiffness matrix and decrease of strain due to shear locking effect).