UNIVERSITAT POLITÈCNICA DE CATALUNYA, BARCELONA

MSC. COMPUTATIONAL MECHANICS ERASMUS MUNDUS

ASSIGNMENT 6: BENDING OF BEAMS

Computational Structural Mechanics & Dynamics

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Part a)

Program in Matlab the Timoshenko 2 nodes beam element with reduce integration for the shear stiffness matrix.

Solution:

The Matlab codes for Euler-Bernoulli and Timoshenko beam theories provided at http://www.cimne.com/mat-fem/beams.asp were used in the implementation of the 2 node Timoshenko beam element with reduce integration. The mat-fem resource also provides us the Matlab files for interaction with GiD.

In general, reduced integration is used to avoid the shear locking effect wherein the singularity of the elemental shear stiffness matrix $\boldsymbol{K}_s^{(e)}$ is ensured by reduced integration for the two node Timoshenko beam element i.e. by using only one integration point, gauss point 1 = gauss point 2 = 0.

The elemental shear stiffness matrix $\boldsymbol{K}_s^{(e)}$ is given as,

$$\boldsymbol{K}_s^{(e)} = \left(\frac{GA^*}{L}\right)^{(e)} \begin{bmatrix} 1 & \frac{l^{(e)}}{2} & -1 & \frac{l^{(e)}}{2} \\ \dots & \frac{(l^{(e)})^2}{4} & -\frac{l^{(e)}}{2} & \frac{(l^{(e)})^2}{4} \\ & \dots & 1 & -\frac{l^{(e)}}{2} \\ Symm & & \dots & \frac{(l^{(e)})^2}{4} \end{bmatrix}$$

This is implemented by modifying the $\boldsymbol{K}_s^{(e)}$ matrix in given code for Timoshenko beam as shown below,

```
% elemental shear stiffness matrix for reduced integration case
   K_s = [  1    ,    len/2 ,   -1    ,    len/2 ;
                  len/2 , len^2/4 , -len/2 , len^2/4 ;
                  -1    ,  -len/2 ,    1    ,  -len/2 ;
                  len/2 , len^2/4 , -len/2 , len^2/4 ];
```

Part b)

Solve the following problem with a 64 element mesh with the

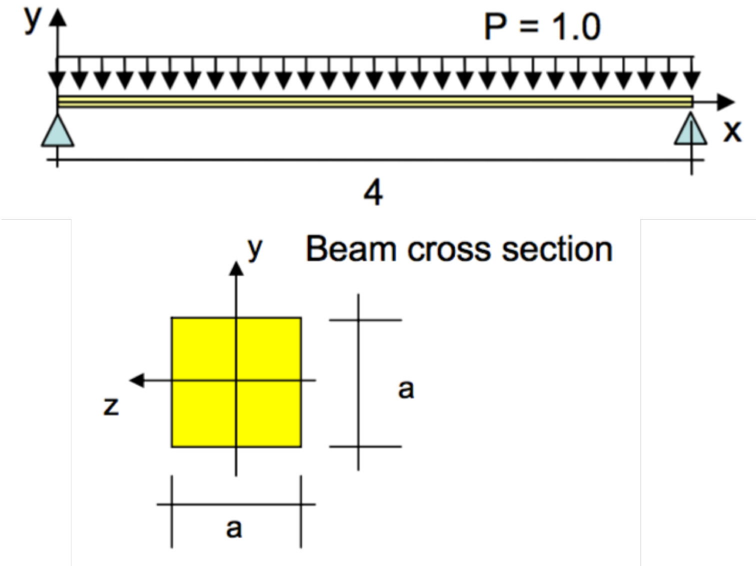


Figure 1

- 2 nodes Euler-Bernoulli element
- 2 nodes Timoshenko Full Integrate element
- 2 nodes Timoshenko Reduce Integration element

Compare maximum displacements, moments and shear for the 3 elements against the a/L relationship.

Solution:

The given simply supported beam of length 4 m is subjected to a uniform load of value 1. Three different types of 2 noded elements are used to discretise the beam into 64 elements. The beam cross-section is varying with a parameter *a* and a comparison study is performed with increasing cross-section area of the beam and bending rigidity *EI*. The following properties of the beam are considered in the analysis using an input file `SimpleSupUL_Beam_64elem.m`,

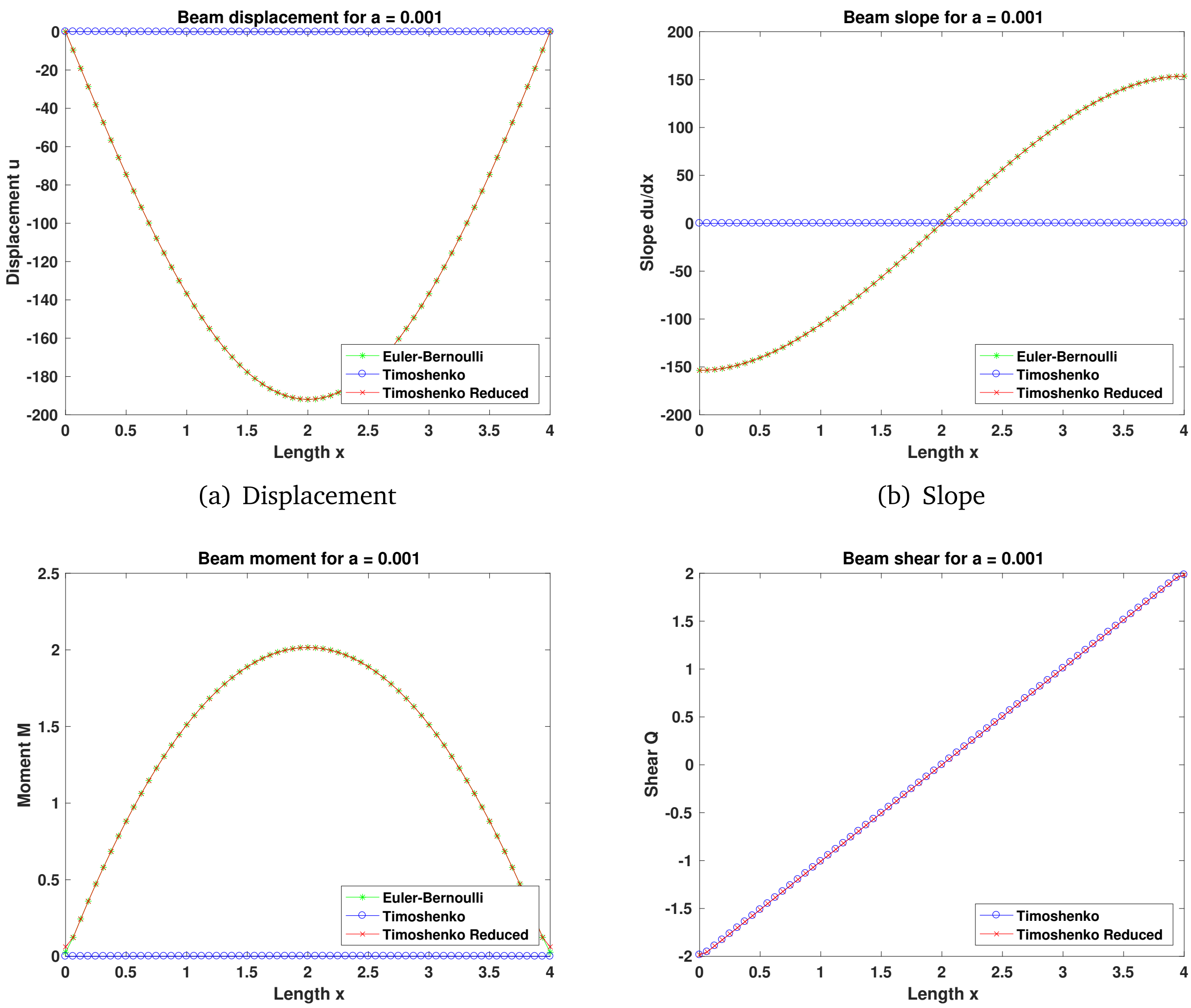
$$E = 210\ e9\ Pa, \quad \nu = 0.25, \quad Area = a^2,$$
$$I = \frac{a^4}{12}, \quad density\ (of\ steel) = 8050\ kg/m^3$$

where a varies with given eight values between 0.001 and 0.4

We are required to plot the displacement, slope, moment and shear of the three different beam element with varying cross-section area of the beam. The Euler-Bernoulli beam theory is based on the assumption that after bending, the plane cross-sections remain plane and perpendicular to its longitudinal axis (bending line) making them stiffer in nature. In case of thick beams, i.e. when the cross-section is comparable with the length of the beam, the shear effect becomes predominant and the beam does not remain perpendicular to the bending line. The theory based on this assumption is called the Timoshenko theory. However, the discretization process of this theory introduces a large error in the finite element method known as shear locking. An approach proposed to overcome this locking is the reduced integration method.

The result plots obtained for the given problem using the provided Matlab files with the three different elements discussed are given below for varying cross-sections of the beam,

For a = 0.001

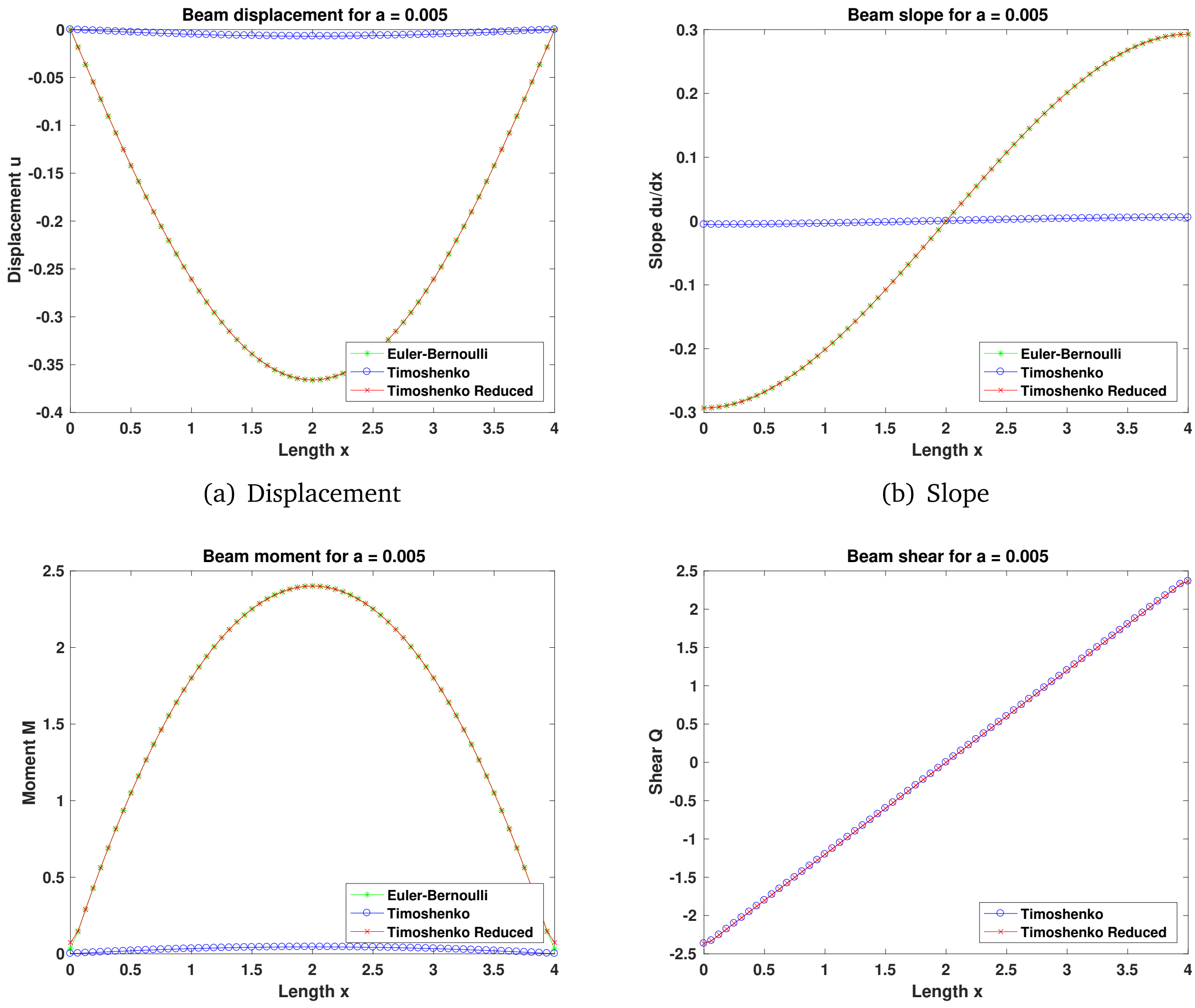


(a) Displacement (b) Slope

(c) Moment (d) Shear

Figure 2: Results obtained with different element types for $a = 0.001$

For a = 0.005



(a) Displacement

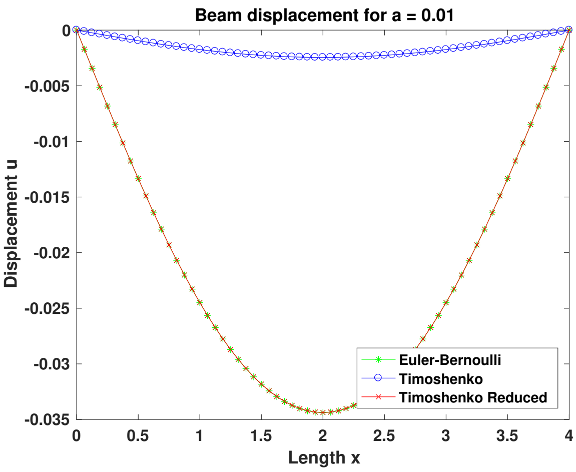
(b) Slope

(c) Moment

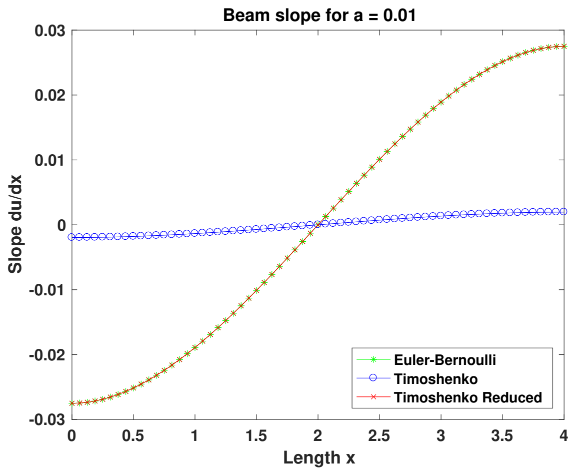
(d) Shear

Figure 3: Results obtained with different element types for $a = 0.005$

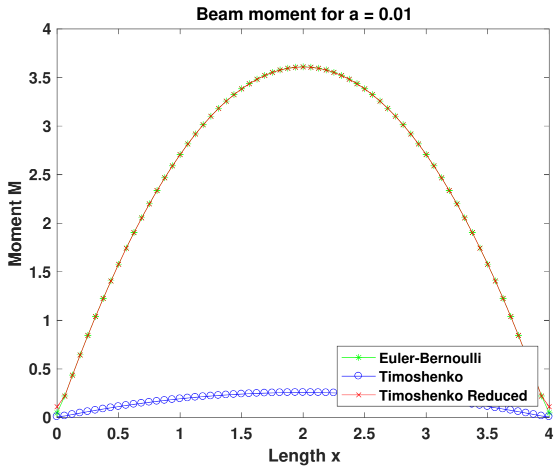
For a = 0.01



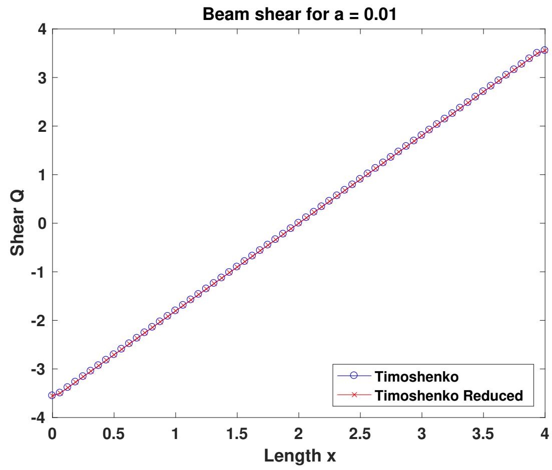
(a) Displacement



(b) Slope



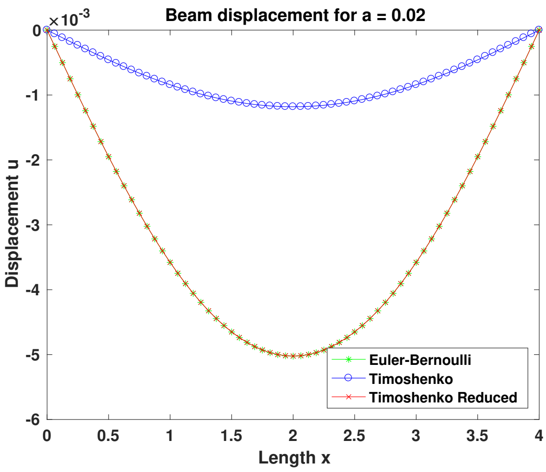
(c) Moment



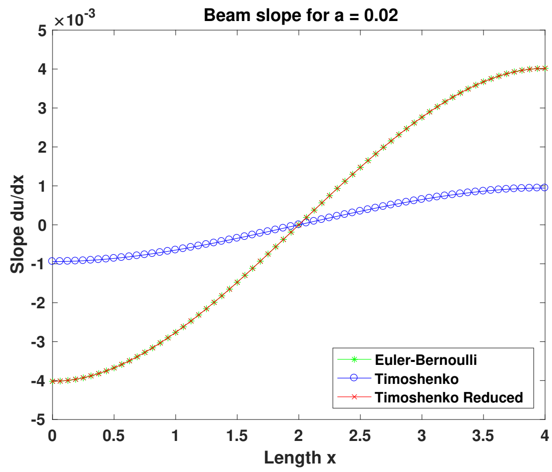
(d) Shear

Figure 4: Results obtained with different element types for $a = 0.01$

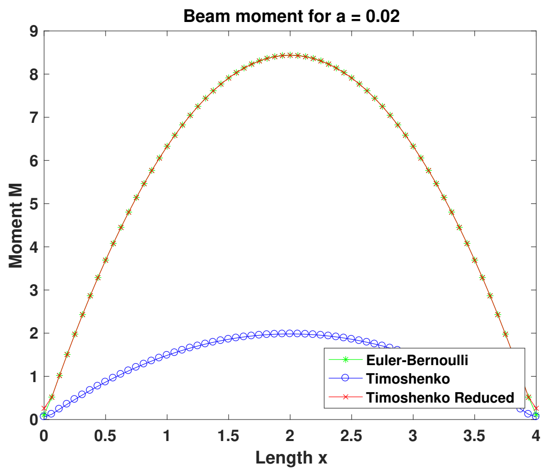
$\boxed{\text{For a = 0.02}}$



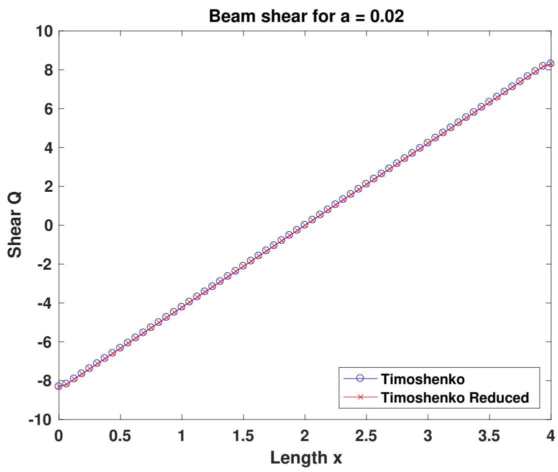
(a) Displacement



(b) Slope



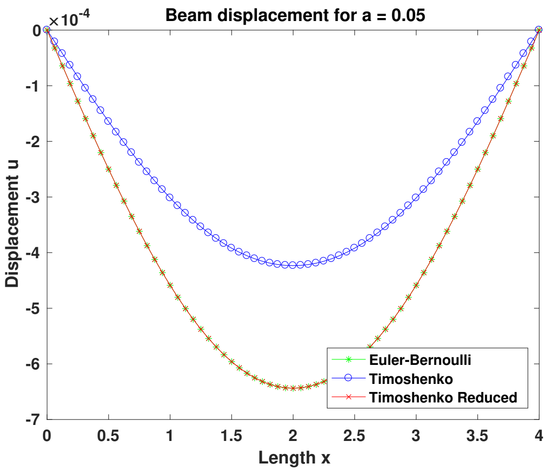
(c) Moment



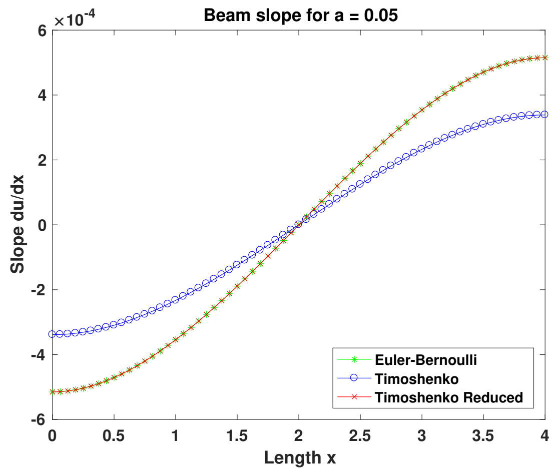
(d) Shear

Figure 5: Results obtained with different element types for $a = 0.02$

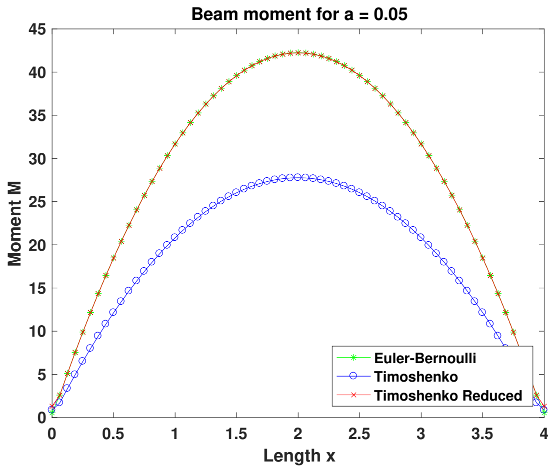
For a = 0.05



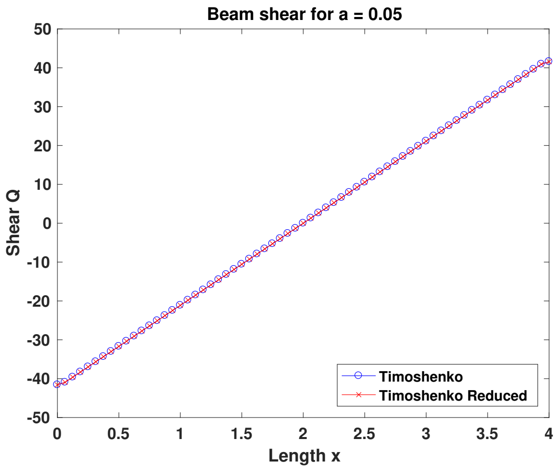
(a) Displacement



(b) Slope



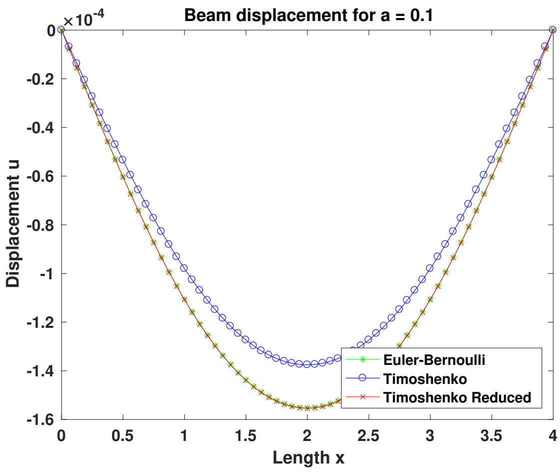
(c) Moment



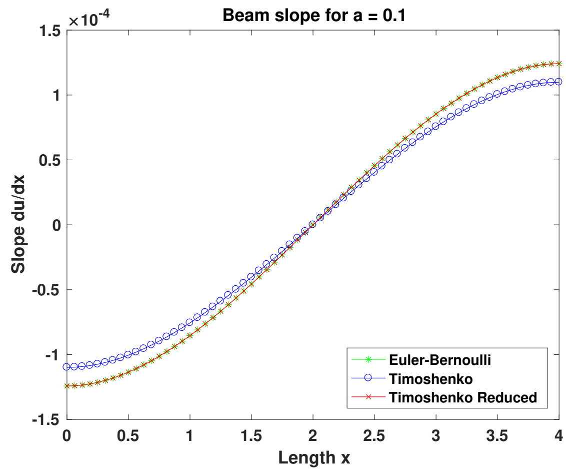
(d) Shear

Figure 6: Results obtained with different element types for $a = 0.05$

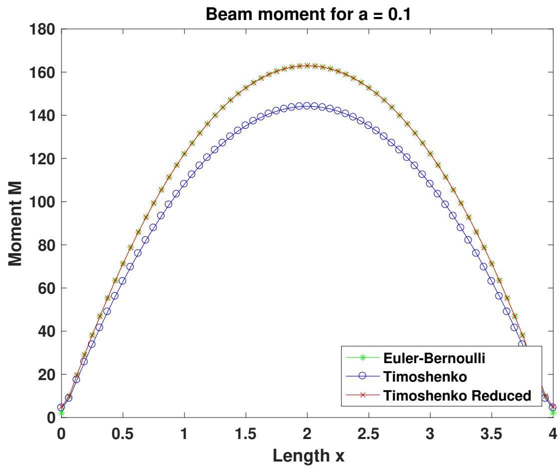
For a = 0.1



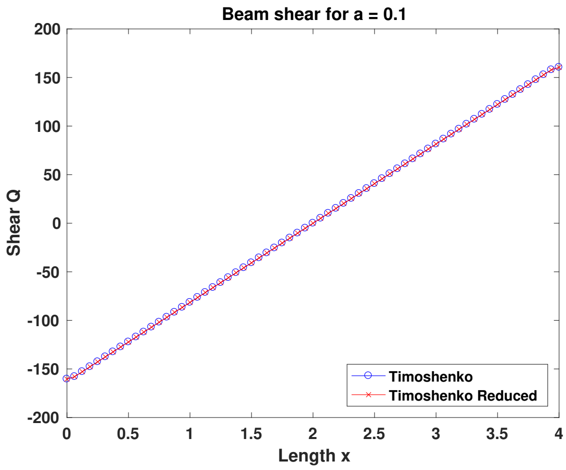
(a) Displacement



(b) Slope



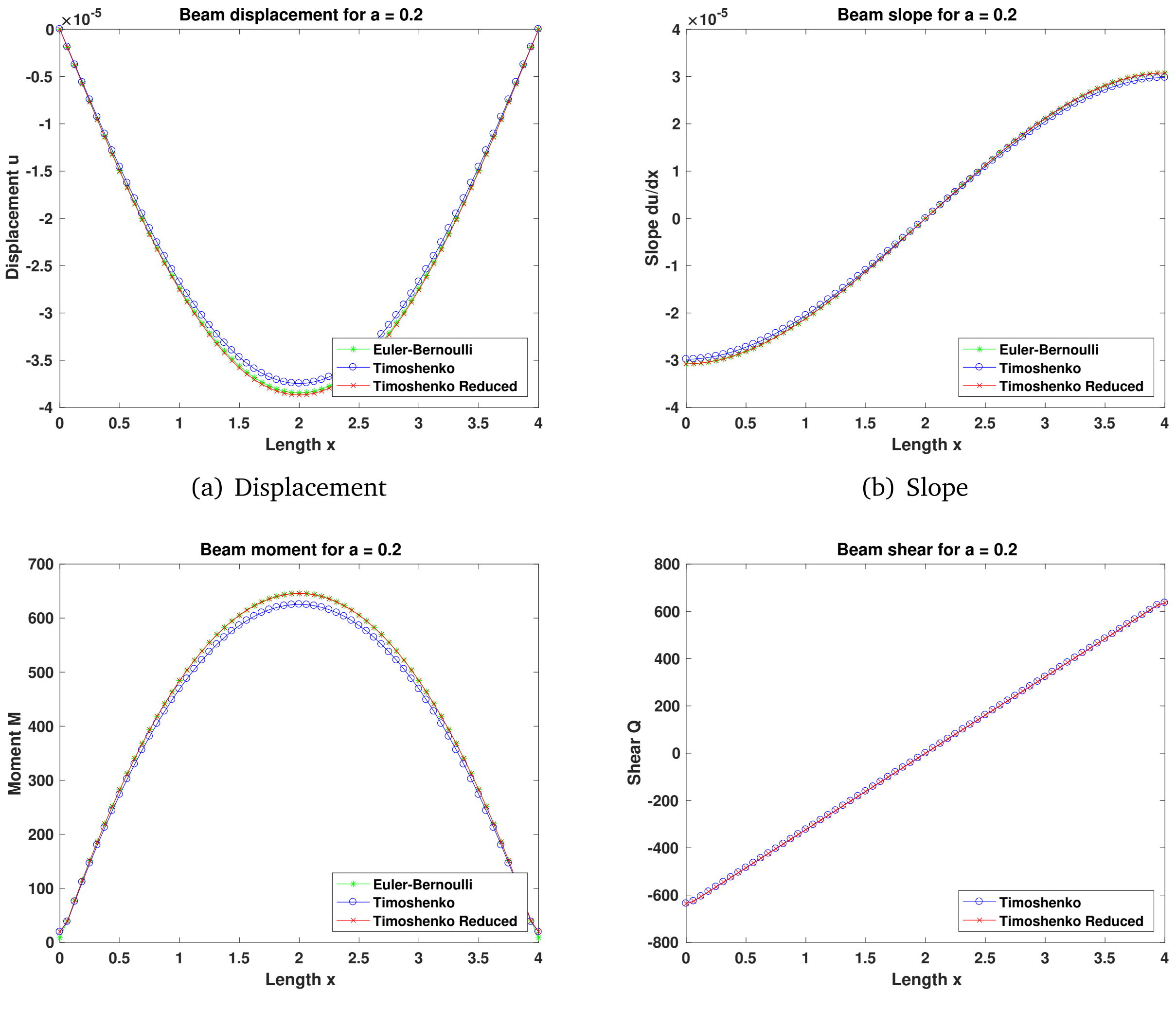
(c) Moment



(d) Shear

Figure 7: Results obtained with different element types for $a = 0.1$

For a = 0.2

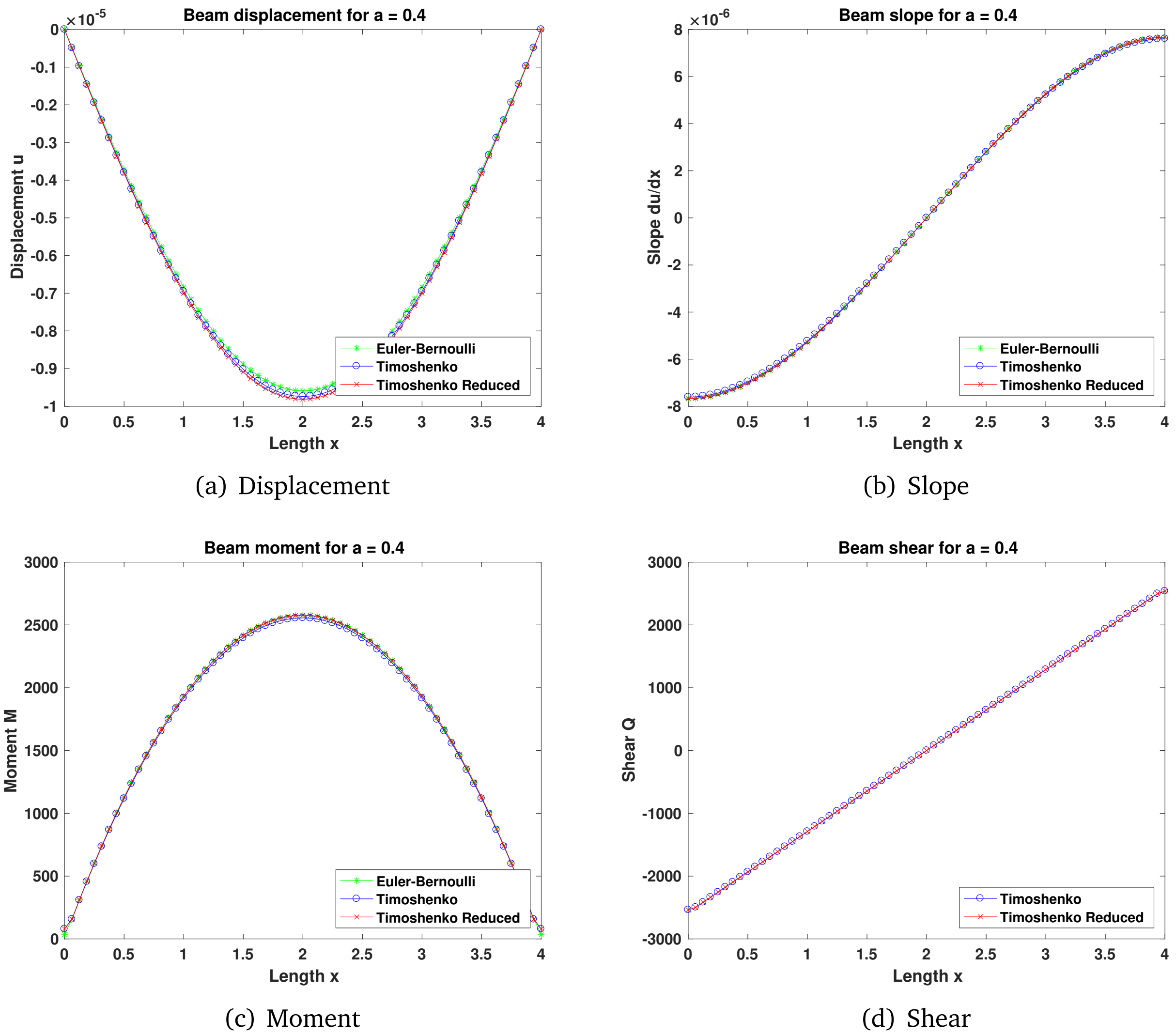


(a) Displacement (b) Slope

(c) Moment (d) Shear

Figure 8: Results obtained with different element types for $a = 0.2$

For a = 0.4



(a) Displacement (b) Slope

(c) Moment (d) Shear

Figure 9: Results obtained with different element types for $a = 0.4$

It is clearly observed from Figures 2-9, as the parameter α increases, the solution for all three elements converges. Essentially, Timoshenko fully integrated element under-predicts the results for smaller cross-section (lower value of α i.e. thin beams) but converges to exact solution for thick beams. A comparison of the maximum values of displacement, slope, moment and shear for all cases is given in Tables 1 and 2.

α	Maximum absolute displacement			Maximum absolute slope		
	Euler-Bernoulli	**Timoshenko full integration**	**Timoshenko reduced integration**	**Euler-Bernoulli**	**Timoshenko full integration**	**Timoshenko reduced integration**
0.001	192.00	0.147	191.93	153.61	0.118	153.57
0.005	0.366	0.007	0.366	0.293	0.006	0.293
0.01	0.034	0.002	0.034	0.028	0.002	0.027
0.02	0.005	0.001	0.005	0.004	9.44 e-4	0.004
0.05	6.44 e-4	4.23 e-4	6.44 e-4	5.15 e-4	3.39 e-4	5.15 e-4
0.1	1.55 e-4	1.38 e-4	1.55 e-4	1.24 e-4	1.10 e-4	1.24 e-4
0.2	3.85 e-5	3.75 e-5	3.87 e-5	3.08 e-5	2.98 e-5	3.08 e-5
0.4	9.59 e-6	9.74 e-6	9.82 e-6	7.67 e-6	7.61 e-6	7.67 e-6

Table 1: Comparison table for results obtained with different element types

α	Maximum absolute moment			Maximum absolute shear	
	Euler-Bernoulli	**Timoshenko full integration**	**Timoshenko reduced integration**	**Timoshenko full integration**	**Timoshenko reduced integration**
0.001	2.016	0.002	2.015	1.984	1.985
0.005	2.402	0.045	2.401	2.365	2.364
0.01	3.610	0.257	3.608	3.554	3.553
0.02	8.440	1.982	8.436	8.308	8.31
0.05	42.248	27.767	42.229	41.590	41.58
0.1	162.993	144.15	162.92	160.45	160.5
0.2	645.972	625.33	645.68	635.91	635.90
0.4	2577.89	2555.94	2576.74	2537.72	2537.71

Table 2: Comparison table for results obtained with different element types continued

In the following figures the logarithm of the maximum values presented in Tables 1 and 2 are plotted for all cases versus the ratio of parameter a/L.

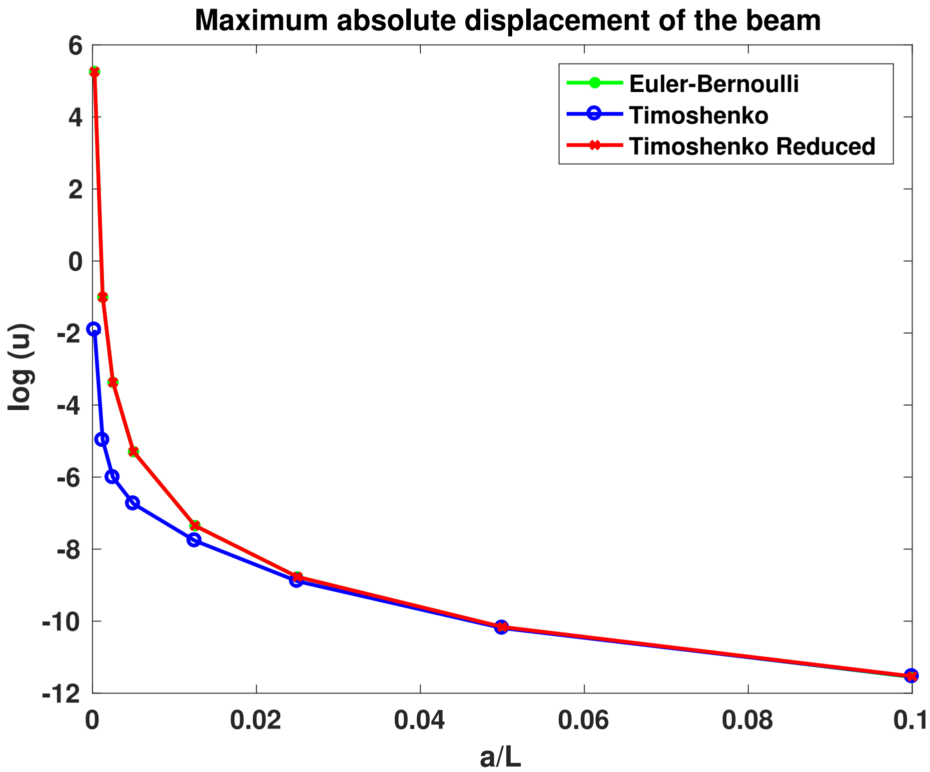


Figure 10: Result plot: logarithm of max displacement vs a/L

It can be inferred from Figures 10 and 11 that for smaller values of a/L i.e. for slender beams, the Timoshenko assumption induces an error in both the displacement and slope of the beam. This effect is neutralised by the use of reduced integration, which gives almost the same result as the Euler-Bernoulli theory.

A similar trend is observed in the plot for moment of the beam in Figure 12. In the full integrated Timoshenko element, rotation between the cross section and the bending line is allowed due to the shear deformation effect. This rotation (shear deformation) is not included in Euler-Bernoulli element which makes it stiffer. It is clear now that Timoshenko full integrated element is not ideal to model the behaviour of bending in slender beams due to the error induced through shear locking. This locking causes extremely low convergence rate towards the exact solution. The thinner the beam, the lower the convergence rate for a Timoshenko beam element as observed in the results obtained from Figures 2-9 Though, for a higher a/L ratio, the beam is less stiffer and the Timoshenko beam element converges with the Euler-Bernoulli solution.

An approach to control this locking problem, the reduced integration technique allows to overcome the influence of the shear stiffness and produces better results for both thick and thin beams, which is illustrated in the results of this analysis shown in Figures 10-12 where the results for reduced integration Timoshenko element is almost same as the results for

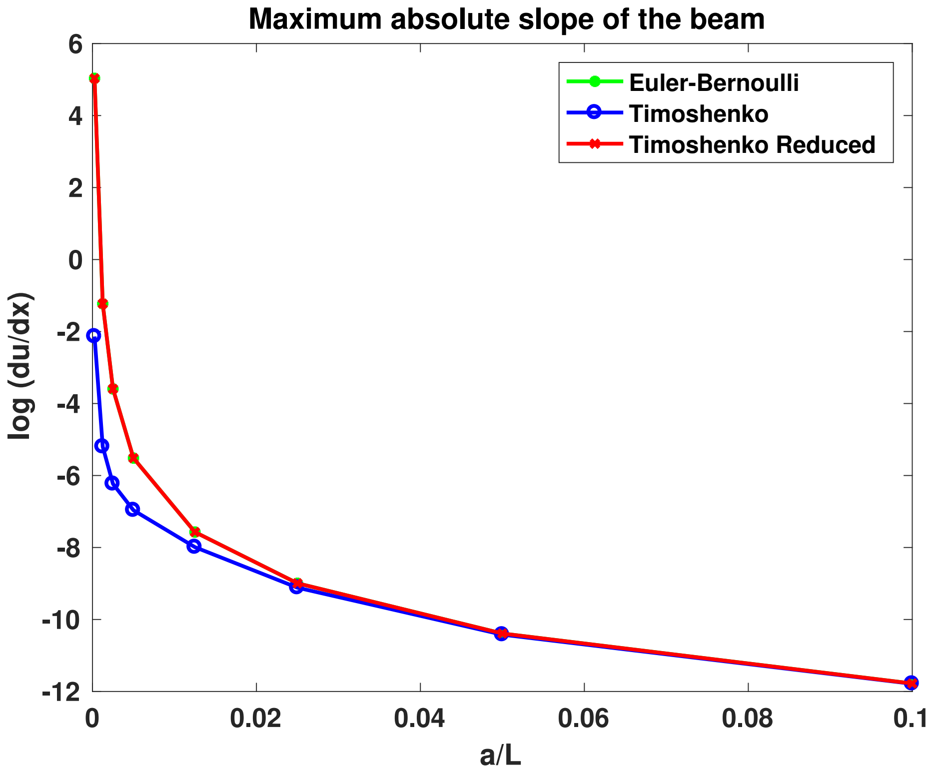


Figure 11: Result plot: logarithm of max slope vs a/L

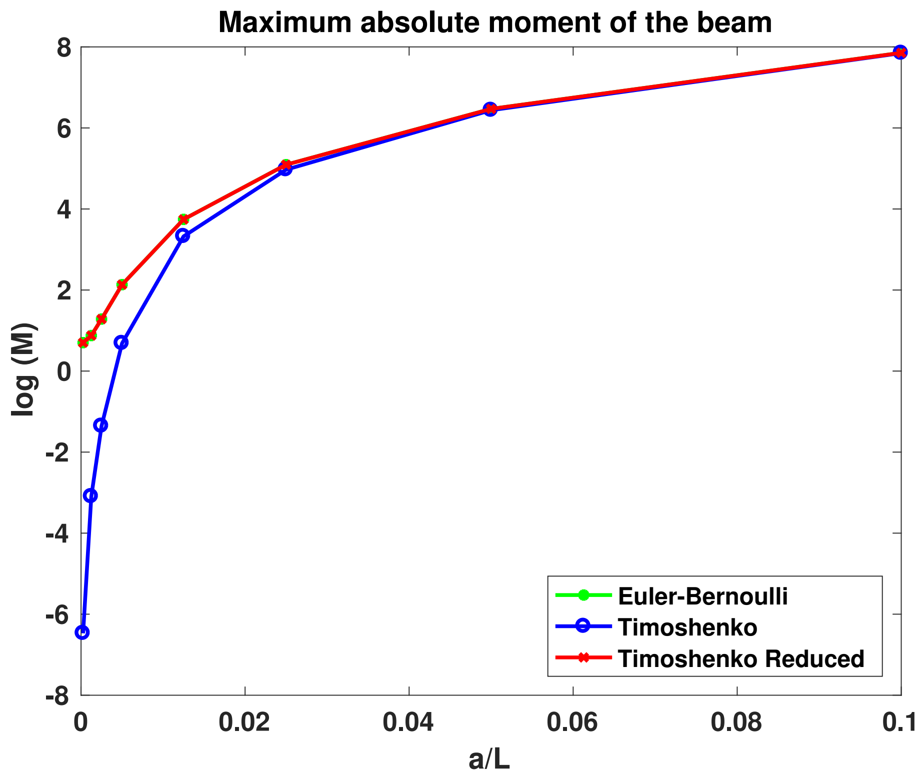


Figure 12: Result plot: logarithm of max moment vs a/L

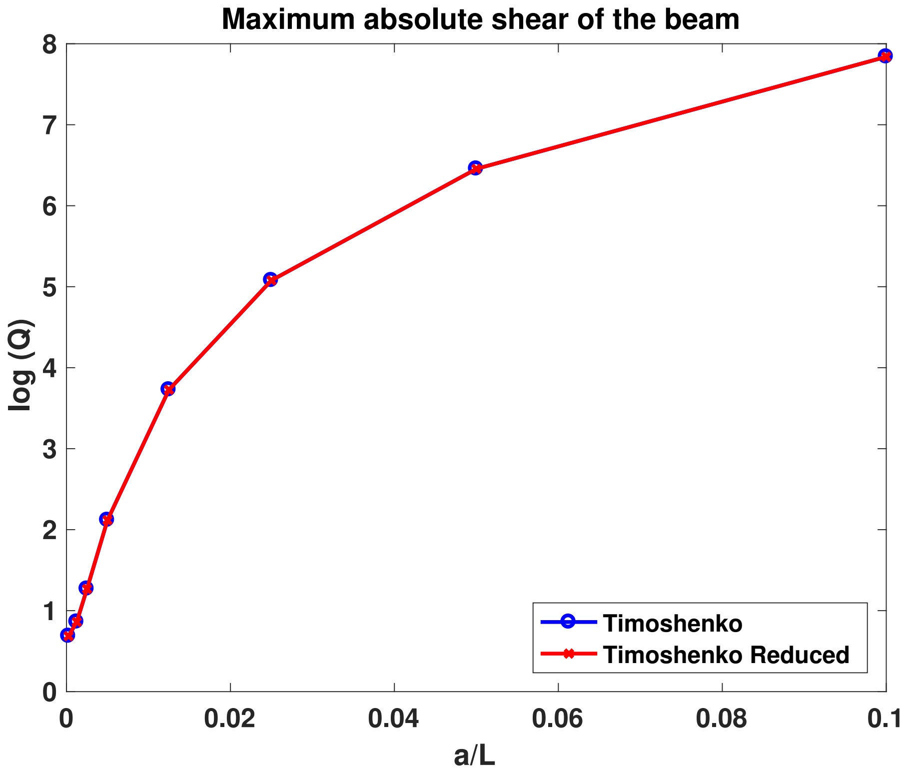


Figure 13: Result plot: logarithm of max shear vs a/L

Euler-Bernoulli element for all values of parameter a (i.e. for both thin and thick beams). Figure 13 presents the results obtained for both Timoshenko full integrated and reduced integrated beams since the Euler-Bernoulli theory neglects the effect of transverse shear deformation. In conclusion, this exercise helped to understand and demonstrate the principles involved in finite element modelling of bending in beams where a complete scientific basis for the locking problems encountered in this field of numerical analysis was provided and the correction criterion using a reduced integration approach was shown for the given problem.