Universitat Politecnica de Catalunya CIMNE
Master in Computational Mechanics

Computational Structural Mechanics & Dynamics

Assignment 9

Dynamics

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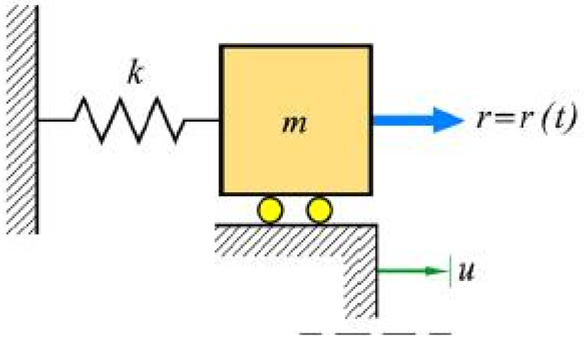
Assignment 9:

1) In the dynamic system below, let r(t) be a constant force F. What is the effect of F on the time-dependent displacement u(t) and the natural frequency of vibration of the system?



According to the Newton's 2nd Law of Motion implied in dynamics,

$$m\ddot{u} + ku = F$$

The solution of this differential equation consists of a general solution (u_g)in case of free undamped vibrations (where F = 0), and a particular solution (u_p) in case of forced undamped vibrations (where F ≠ 0).

Solution to the above force balance equation can be read as, $u(t) \; = u_g(t) + u_p(t)$

For the general solution, putting F = 0, we get : $m\ddot{u} + ku = 0$

$$\therefore \; u_g(t) = A \sin \omega t + B \cos \omega t$$

where, $\omega = \sqrt{\frac{k}{m}}$

For the particular solution, putting F ≠ 0, we get : $ku = F$

$$\therefore \; u_p(t) = \frac{F}{k}$$

The total solution of the case can be read as,

$$u(t) \; = A \sin \omega t + B \cos \omega t + \frac{F}{k}$$

Considering the initial conditions at t = 0, $u(t) = 0$

We get, A = 0 and B = - $\frac{F}{k}$

Substituting back in the displacement equation, we get the solution of our differential equation,

$$\boxed{u(t) = \frac{F}{k}(1 - \cos \omega t)}$$

From the above equation, we can say that the displacement would depend on the value of F, but the natural frequency of vibration would remain unaffected with respect to F.

2) A weight whose mass is m is placed at the middle of a uniform axial bar of length L that is clamped at both ends. The mass of the bar may be neglected. Estimate the natural frequency of vibration in terms of m, L, E and A.
Suggestion: First determine the effective k.

As the bar is clamped at both ends with the mass placed in the middle, the force acting on the bar is along the dof in the negative vertical direction. The maximum displacement would occur at the middle of the bar due to the effect of gravity.

The natural frequency of vibration ω is given by $\sqrt{\frac{k}{m}}$, and we know that $F = ku$.

The maximum displacement at the centre of the uniform axial bar is given by,

$$u = \frac{FL^3}{192EI}$$

With

- F = mg
- Considering square cross section, I = $\frac{l^4}{12}$ where l is the side of the square.

$$\therefore u = \frac{mgL^3}{192E\,\frac{l^4}{12}} = \frac{mgL^3}{16El^4}$$

Now, substituting values F = mg and u in $F = ku$

$$\therefore mg = k\frac{mgL^3}{16El^4}$$

$$\therefore k = \frac{16El^4}{L^3}$$

$$\therefore \omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{16El^4}{mL^3}}$$

3) Use the expression on slide 18 to derive the mass matrix of slide 17.

Given: $\boldsymbol{m = \int_0^l N^T N\, \rho dV}$

To derive: $\boldsymbol{m} = \begin{bmatrix} \frac{\rho AL}{3} & \frac{\rho AL}{6} \\ \frac{\rho AL}{6} & \frac{\rho AL}{3} \end{bmatrix}$

The shape function matrix is given by $N = [N_1 \quad N_2]$

For a linear 1D element the shape functions are,

$$N_1 = 1 - \frac{x}{l} \quad and \quad N_2 = \frac{x}{l}$$

$$\therefore m = \int_0^l \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} [N_1 \quad N_2] \, \rho A \, dx$$

$$\therefore m = \rho A \int_0^l \begin{bmatrix} N_1^2 & N_1 N_2 \\ N_2 N_1 & N_2^2 \end{bmatrix} dx$$

$$\therefore m = \rho A \int_0^l \begin{bmatrix} \left(1 - \frac{x}{l}\right)^2 & \left(1 - \frac{x}{l}\right)\frac{x}{l} \\ \left(1 - \frac{x}{l}\right)\frac{x}{l} & \left(\frac{x}{l}\right)^2 \end{bmatrix} dx$$

Solving the integral we get,

$$\therefore m = \begin{bmatrix} \frac{\rho AL}{3} & \frac{\rho AL}{6} \\ \frac{\rho AL}{6} & \frac{\rho AL}{3} \end{bmatrix}$$

4) Obtain also the mass matrix of a two-node, linear displacement element with a variable cross-sectional area that varies from A1 to A2.

As we had used a linear element with 2 nodes for solving the previous question, we can use the mass matrix obtained earlier in order to evaluate these results.

Thus, the linear interpolation that can be used for this varying area is,

$$A(x) = A_1 N_1(x) + A_2 N_2(x)$$

The mass matrix therefore becomes after substitution in the formula,

$$\therefore m = \rho \int_0^l A_1 N_1 \begin{bmatrix} N_1^2 & N_1 N_2 \\ N_2 N_1 & N_2^2 \end{bmatrix} dx + \rho \int_0^l A_2 N_2 \begin{bmatrix} N_1^2 & N_1 N_2 \\ N_2 N_1 & N_2^2 \end{bmatrix} dx$$

$$\therefore m = \rho \int_0^l A_1 \begin{bmatrix} N_1^3 & N_1^2 N_2 \\ N_2 N_1^2 & N_2^3 \end{bmatrix} dx + \rho \int_0^l A_2 \begin{bmatrix} N_1^3 & N_1^2 N_2 \\ N_2 N_1^2 & N_2^3 \end{bmatrix} dx$$

$$\therefore m = A_1 \begin{bmatrix} \frac{\rho L}{4} & \frac{\rho L}{12} \\ \frac{\rho L}{12} & \frac{\rho L}{12} \end{bmatrix} + A_2 \begin{bmatrix} \frac{\rho L}{12} & \frac{\rho L}{12} \\ \frac{\rho L}{12} & \frac{\rho L}{4} \end{bmatrix}$$

5) A uniform two-node bar element is allowed to move in a 3D space. The nodes have only translational d.o.f. What is the diagonal mass matrix of the element?

As the 2 node bar moves in 3D space, each node will have 3 dof. Thus, making it a 6x6 matrix. Now, the given condition is that, it has only translational dof while the rotations are not there. In the mass matrix, the diagonal terms represent the displacements while the other terms in the matrix represent the rotations.

Thus, accordingly since we have only translations and no rotations, the mass matrix is given by,

$$\therefore m = \frac{\rho A L}{2}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$