Computational Structural Mechanics & Dynamics

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M.Sc. computational Mechanics (2016-2018)

Assignment 4

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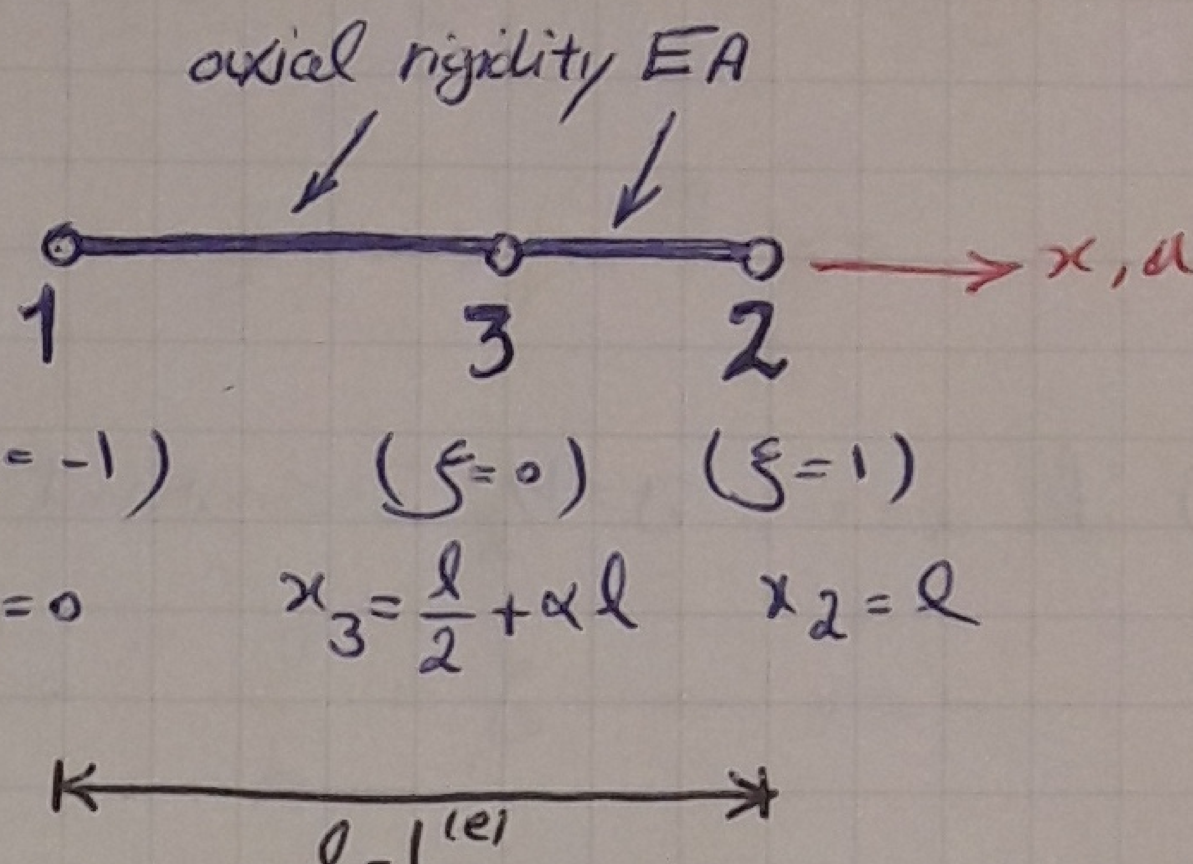
Assignment 4.1

$$\begin{bmatrix} 1 \\ x \\ u \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ N_3^e \end{bmatrix}$$

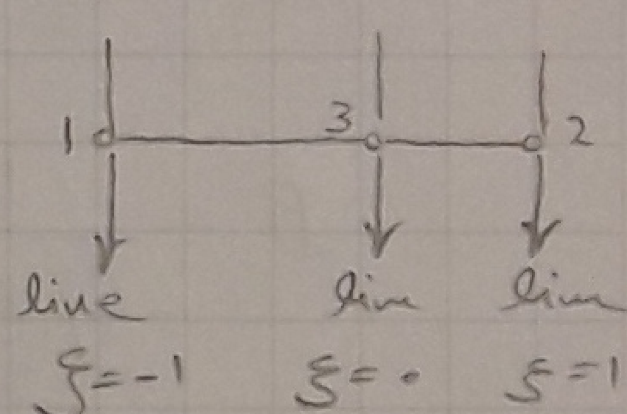


$$-\frac{1}{2} < \alpha < \frac{1}{2}$$

1- Jacobian?

2nd equation: $x = x_1 N_1^e + x_2 N_2^e + x_3 N_3^e = \sum_{i=1}^{3} x_i N_i^e$

$$\begin{cases} N_1^e = (\xi-0)(\xi-1)(\frac{1}{2}) = \frac{1}{2}(\xi^2-\xi) \\ N_2^e = (\xi+1)(\xi-0)(\frac{1}{2}) = \frac{1}{2}(\xi^2+\xi) \\ N_3^e = (\xi+1)(\xi-1)(-1) = (1-\xi^2) \end{cases}$$



$$x = x_1 \cdot \frac{1}{2}(\xi^2-\xi) + x_2 \cdot \frac{1}{2}(\xi^2+\xi) + x_3 \cdot (1-\xi^2)$$

$$J = \frac{dx}{d\xi} = \frac{1}{2}x_1(2\xi-1) + \frac{1}{2}x_2(2\xi+1) + x_3(-2\xi)$$

$$J = x_1(\xi-\tfrac{1}{2}) + x_2(\xi+\tfrac{1}{2}) - 2x_3\xi \quad \left| \begin{array}{l} x_1=0 \\ x_2=l \\ x_3=l(\frac{1}{2}+\alpha) \end{array} \right.$$

$$J = 0 + l(\xi+\tfrac{1}{2}) - l(1+2\alpha)\xi =$$

$$= \cancel{l\xi} + l/2 - \cancel{l\xi} - 2\alpha l\xi \Rightarrow \boxed{J = l(\frac{1}{2} - 2\alpha\xi)}$$

a) if $-\frac{1}{4} < \alpha < \frac{1}{4}$ then $J > 0$ over the whole element $-1 \leq \xi \leq 1$

$$J = l\left(\frac{1}{2} - 2\alpha\xi\right)$$

$$\xi = 1 \longrightarrow J = l\left(\frac{1}{2} - 2\alpha\right) > 0 \Rightarrow \frac{1}{2} - 2\alpha > 0 \Rightarrow \alpha < \frac{1}{4}$$

$$\xi = -1 \longrightarrow J = l\left(\frac{1}{2} + 2\alpha\right) > 0 \Rightarrow \frac{1}{2} + 2\alpha > 0 \Rightarrow \alpha > -\frac{1}{4}$$

$$\xi = 0 \longrightarrow J = l\left(\frac{1}{2}\right) > 0$$

$\Rightarrow$ if $-\frac{1}{4} < \alpha < \frac{1}{4} \Rightarrow$ over $-1 \leq \xi \leq 1$ $J > 0$

b) if $\alpha = 0$, $J = 1/2$ is constant over element

$$\alpha = 0 \longrightarrow J = \frac{l}{2} \quad \text{constant } \checkmark$$

2- $B = \frac{dN}{dx} = J^{-1}\frac{dN}{d\xi}$, $N = [N_1 \; N_2 \; N_3]$ $\qquad e = \frac{du}{dx} = B\underline{u}^e$

$$N = \left[\frac{1}{2}(\xi^2 - \xi) \quad \frac{1}{2}(\xi^2 + \xi) \quad (1 - \xi^2)\right]$$

$$\frac{dN}{d\xi} = \left[\xi - \frac{1}{2} \quad \xi + \frac{1}{2} \quad -2\xi\right]$$

$$B = \frac{1}{l\left(\frac{1}{2} - 2\alpha\xi\right)}\left[\xi - \frac{1}{2} \quad \xi + \frac{1}{2} \quad -2\xi\right]$$

3- $K^e = \int_0^l EAB^TB\,dx = \int_{-1}^{1} EAB^TBJ\,d\xi$ $\qquad$ Element Stiffness Matrix

$$EA\int_0^l B^TB\,dx = EA\int_0^l \frac{dN^T}{dx}\frac{dN}{dx}\,dx =$$

$$= EA\int_{-1}^{1} J^{-1}\frac{dN^T}{d\xi}\,J\,\frac{dN}{d\xi}\,J\,d\xi =$$

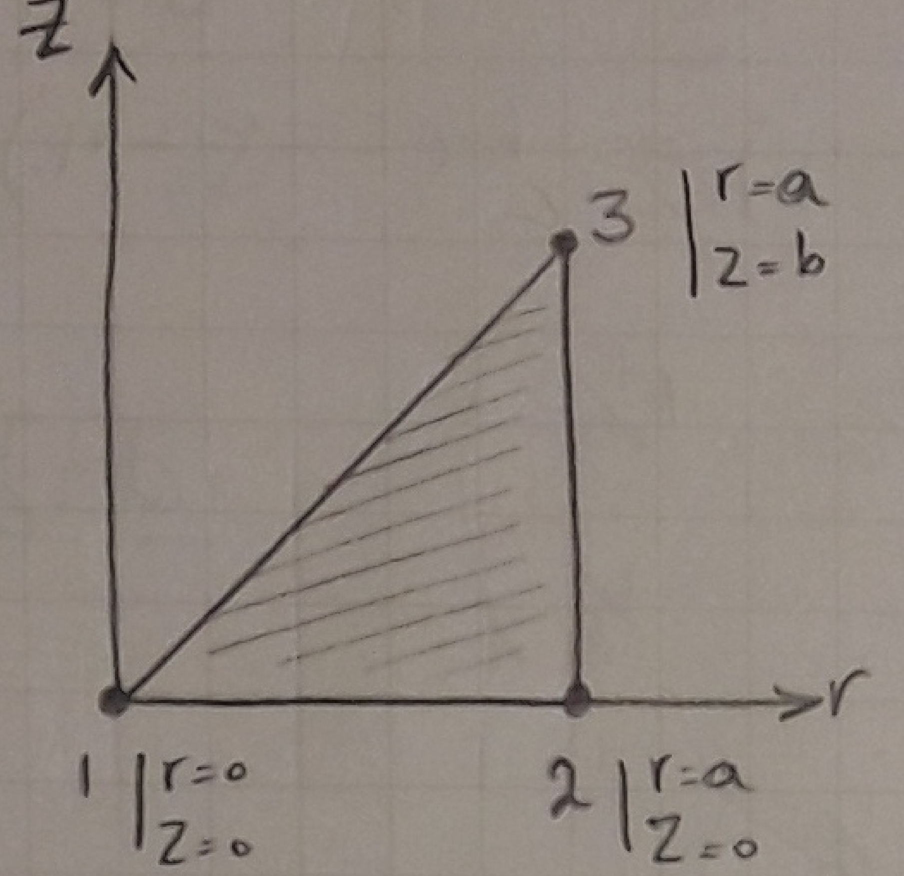
$$= EA\int_{-1}^{1} B^TB\,J\,d\xi$$

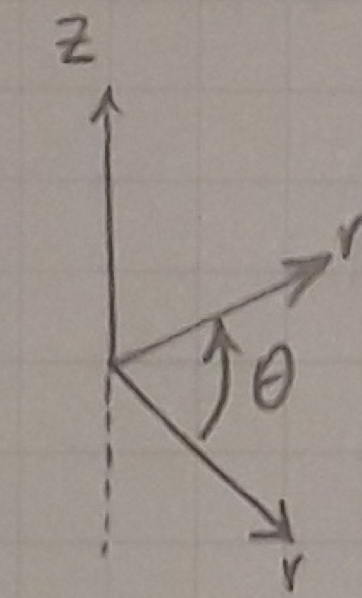
① compute K^e

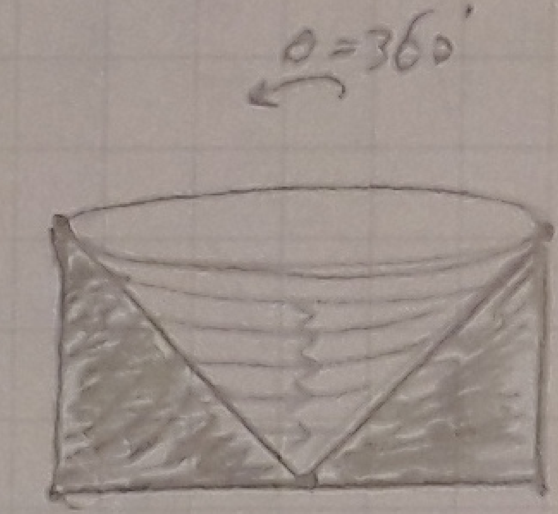
$r_1 = 0$ $\quad$ $z_1 = z_2 = 0$ $\quad$ $\nu = 0$

$r_2 = r_3 = a$ $\quad$ $z_3 = b$

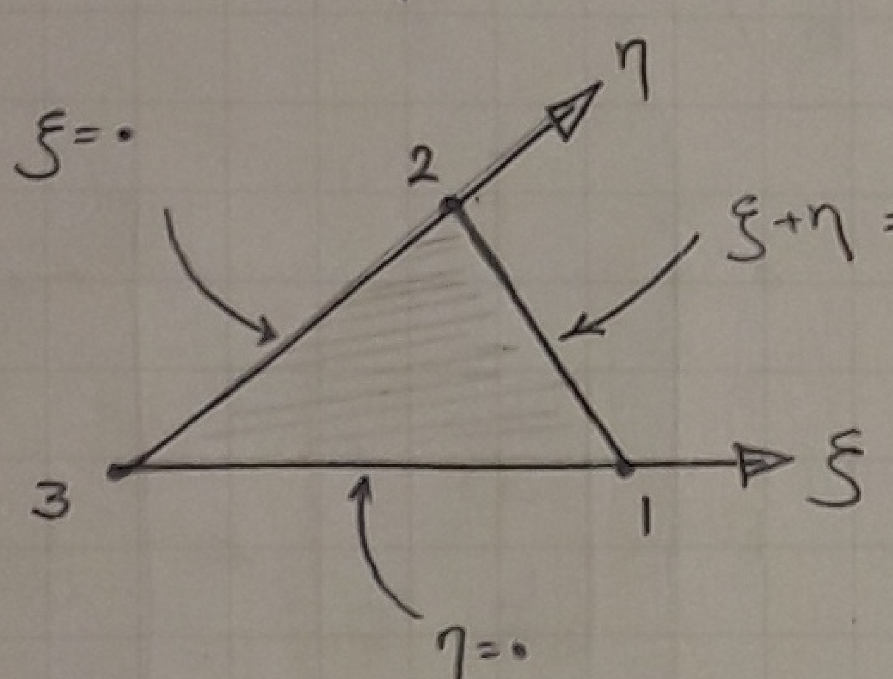
$$\underline{E} = E \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$







* change from Triangular to natural coordinate...



=>

#	(ξ, η)
1	(1, 0)
2	(0, 1)
3	(0, 0)

$$\begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} = \begin{bmatrix} \xi \\ \eta \\ 1-(\xi+\eta) \end{bmatrix}$$

each node has 2 D.o.F.

$$D = \begin{bmatrix} \frac{\partial}{\partial r} & 0 \\ 0 & \frac{\partial}{\partial z} \\ \frac{1}{r} & 0 \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial r} \end{bmatrix}_{4\times 2}$$

$$N = \left[\begin{array}{cc|cc|cc} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{array}\right]_{2\times 6}$$

$$B = \left[\begin{array}{c|c|c} B_1 & B_2 & B_3 \end{array}\right]_{4\times 6}$$

$$B_i = \begin{bmatrix} q_r & 0 \\ 0 & q_z \\ q_\theta & \\ q_z & q_r \end{bmatrix}_i = \begin{bmatrix} \frac{\partial N_i}{\partial r} & 0 \\ 0 & \frac{\partial N_i}{\partial z} \\ \frac{N_i}{r} & 0 \\ \frac{\partial N_i}{\partial z} & \frac{\partial N_i}{\partial r} \end{bmatrix}_{4\times 2}$$

Note: For this Matrix we can even use alternative dof's arrangment like |page 14| of slides; but in this case part 2 of anignment 4.2 would not be consistent with problem!

$$J = \frac{\partial(r,z)}{\partial(\xi,\eta)} = \begin{pmatrix} \frac{\partial r}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial r}{\partial \eta} & \frac{\partial z}{\partial \eta} \end{pmatrix}$$

$$r(\xi,\eta) = \sum_1^3 r_i N_i(\xi,\eta) = 0 + a\eta + a(1-\xi-\eta) = a(1-\xi)$$

$$z(\xi,\eta) = \sum_1^3 z_i N_i(\xi,\eta) = 0 + 0 + b(1-\xi-\eta) = b(1-\xi-\eta)$$

$$\Rightarrow J = \begin{pmatrix} -a & -b \\ 0 & -b \end{pmatrix} \longrightarrow J^{-1} = \begin{pmatrix} -1/a & +1/a \\ 0 & -1/b \end{pmatrix} \qquad \underline{J \cdot J^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \checkmark$$

(INVERT) $$J^{-1} = \frac{\partial(\xi,\eta)}{\partial(r,z)} = \begin{pmatrix} \partial\xi/\partial r & \partial\eta/\partial r \\ \partial\xi/\partial z & \partial\eta/\partial z \end{pmatrix} = \begin{pmatrix} -1/a & 1/a \\ 0 & -1/b \end{pmatrix}$$

$$\underset{3\times2}{\begin{bmatrix} \frac{\partial N_1}{\partial r} & \frac{\partial N_1}{\partial z} \\ \frac{\partial N_2}{\partial r} & \frac{\partial N_2}{\partial z} \\ \frac{\partial N_3}{\partial r} & \frac{\partial N_3}{\partial z} \end{bmatrix}} = \underset{3\times2}{\begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \eta} \\ \frac{\partial N_2}{\partial \xi} & \frac{\partial N_2}{\partial \eta} \\ \frac{\partial N_3}{\partial \xi} & \frac{\partial N_3}{\partial \eta} \end{bmatrix}} \underset{2\times2}{\begin{bmatrix} \frac{\partial \xi}{\partial r} & \frac{\partial \xi}{\partial z} \\ \frac{\partial \eta}{\partial r} & \frac{\partial \eta}{\partial z} \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} -\frac{1}{a} & 0 \\ \frac{1}{a} & -\frac{1}{b} \end{bmatrix} = \begin{bmatrix} -\frac{1}{a} & 0 \\ \frac{1}{a} & \frac{-1}{b} \\ 0 & \frac{1}{b} \end{bmatrix}$$

$$\frac{N1}{r} = \frac{\xi}{a(1-\xi)} \xrightarrow{\xi=\eta=1/3} \frac{1}{2a}$$

$$\frac{N2}{r} = \frac{\eta}{a(1-\xi)} \xrightarrow{\xi=\eta=1/3} \frac{1}{2a}$$

$$\frac{N3}{r} = \frac{1-\xi-\eta}{a(1-\xi)} \xrightarrow{\xi=\eta=1/3} \frac{1}{2a}$$

$$B = \begin{bmatrix} -\frac{1}{a} & 0 & \frac{1}{a} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{b} & 0 & \frac{1}{b} \\ \frac{1}{2a} & 0 & \frac{1}{2a} & 0 & \frac{1}{2a} & 0 \\ 0 & -\frac{1}{a} & -\frac{1}{b} & \frac{1}{a} & \frac{1}{b} & 0 \end{bmatrix}$$

$$K = \int_z \int_r \int_0^{\theta} B^T E B \; r \, d\theta \, dr \, dz$$

$$B^T E B = \begin{bmatrix} -\frac{1}{a} & 0 & \frac{1}{2a} & 0 \\ 0 & 0 & 0 & -\frac{1}{a} \\ \frac{1}{a} & 0 & \frac{1}{2a} & -\frac{1}{b} \\ 0 & -\frac{1}{b} & 0 & \frac{1}{a} \\ 0 & 0 & \frac{1}{2a} & \frac{1}{b} \\ 0 & \frac{1}{b} & 0 & 0 \end{bmatrix}_{6\times 4} \begin{bmatrix} E & 0 & 0 & 0 \\ 0 & E & 0 & 0 \\ 0 & 0 & E & 0 \\ 0 & 0 & 0 & \frac{E}{2} \end{bmatrix} \begin{bmatrix} -\frac{1}{a} & 0 & \frac{1}{a} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{b} & 0 & \frac{1}{b} \\ \frac{1}{2a} & 0 & \frac{1}{2a} & 0 & \frac{1}{2a} & 0 \\ 0 & -\frac{1}{a} & -\frac{1}{b} & \frac{1}{a} & \frac{1}{b} & 0 \end{bmatrix}$$

$$K^e = \iint_A B^T E B \, (r \, \overset{2\pi}{\cancel{\theta}}) \underbrace{dr\,dz}_{dA} = \sum_{j=1}^{intg.} \sum_{i=1}^{intg.} \Big[B^T E B \, (2\pi r) \, |J| \Big]_{\xi_i, \eta_j} (w_i * w_j)$$

$\#$ intg. point $= 1$ $\qquad \xi_1 = \eta_1 = \frac{1}{3}$, $\; w_1 = 1$

$* \; r = a(1-\xi) = \frac{2}{3} a \qquad * \; |J| = ab \qquad K^e = \overset{\frac{4}{3}\pi a^2 b}{\cancel{2\pi \left(\frac{2}{3} a\right)(ab)}} \left[B^T E B \right]$

$$K^{(e)} = \frac{4}{3} \pi a^2 b \cdot E \begin{bmatrix} \frac{5}{4a^2} & 0 & \frac{-3}{4a^2} & 0 & \frac{1}{4a^2} & 0 \\ 0 & \frac{1}{2a^2} & \frac{1}{2ab} & \frac{-1}{2a^2} & \frac{-1}{2ab} & 0 \\ \frac{-3}{4a^2} & \frac{1}{2ab} & \frac{5}{4a^2} + \frac{1}{2b^2} & \frac{-1}{2ab} & \frac{1}{4a^2} - \frac{1}{2b^2} & 0 \\ 0 & \frac{-1}{2a^2} & \frac{-1}{2ab} & \frac{1}{b^2} + \frac{1}{2a^2} & \frac{1}{ab} & \frac{-1}{b^2} \\ \frac{1}{4a^2} & \frac{-1}{2ab} & \frac{1}{4a^2} - \frac{1}{2b^2} & \frac{1}{2ab} & \frac{1}{4a^2} + \frac{1}{2b^2} & 0 \\ 0 & 0 & 0 & \frac{-1}{b^2} & 0 & \frac{1}{b^2} \end{bmatrix}_{6\times 6} \begin{array}{l} \\ \rightarrow \text{Sum}=0 \\ \\ \rightarrow \text{Sum}=0 \\ \\ \rightarrow \text{Sum}=0 \end{array}$$

part 2: a) Sum of $\frac{\text{row}}{\text{col}}$ 2-4-6 of K^e is vanished. ✓

b) Sum of $\frac{\text{row}}{\text{col}}$ 1-3-5 of K^e is not vanished. ✓

physical interpretation

$$\begin{bmatrix} K_{11} & K_{12} & \cdots & K_{16} \\ K_{21} & K_{22} & \cdots & K_{26} \\ \vdots & & & \\ K_{61} & K_{62} & \cdots & K_{66} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{6} K_{1j} \\ \sum_{j=1}^{6} K_{2j} \\ \vdots \\ \sum_{j=1}^{6} K_{6j} \end{bmatrix} \begin{array}{l} \to \neq 0 \\ \to = 0 \\ \\ \\ \end{array}$$

* this deformation vector is showing a rigid Motion movement in Vertical direction (Z) + a radial deformation in direction (r)
* due to the fact that we do not anticipate any force or stress regarding rigid motion in Z direction so f_2, f_4, f_6 corresponding to Z d.o.f. must be zero, so $\sum_{j=1}^{6} K_{ij}$, $i = 2,4,6 = 0$ ✓
* Since we know that we would have a non zero stress in radial deformation, so $f_i = \sum_{j=1}^{6} K_{ij}$ $i = 1,3,5 \neq 0$ ✓

part 3: force vector $\underline{f}^e$ gravity forces $b = [0, -g]^T$

$$b(r,z) = \begin{pmatrix} b_r(r,z) = 0 \\ b_z(r,z) = -g \end{pmatrix}$$

(we use 1 integration point) $\to \xi = \eta = \frac{1}{3}$
$W = 1$

$$f_{ext} = \sum_{1}^{int.} \sum_{1}^{int.} w_i w_j \underline{N}^T \underline{b}\, r\, J = (1) \times (1) \times \underline{N}^T \times \underline{b} \times \left(\frac{2a}{3}\right) \times (ab)$$

$$N = \begin{bmatrix} \xi & 0 & \eta & 0 & 1-\xi-\eta & 0 \\ 0 & \xi & 0 & \eta & 0 & 1-\xi-\eta \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix}$$

$$f_{ext} = \underset{6\times 2}{N^T} \begin{pmatrix} 0 \\ -g \end{pmatrix} \times \frac{2a^2 b}{3} = \frac{2}{9} a^2 b \begin{pmatrix} 0 \\ -g \\ 0 \\ -g \\ 0 \\ -g \end{pmatrix}$$