CSMD: Assignment 6

Xavier Corbella Coll March 17, 2016

Assignment 6.1

The meshes displayed in figure 1 fail interelement compatibility. For cases a) b) and c), this incompatibility happens because some nodes do not match. In d) e) and f), the nodes match but there are violations of the interelemental continuity due to the different order of the shape functions.

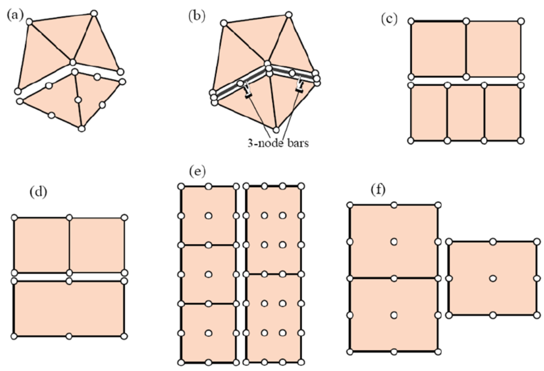

Figure 1

Assignment 6.2

The shape functions of the 3-node bar element (figure 2) are quadratic polynomials in ξ :

$$N_1^e(\xi) = \frac{1}{2}(\xi)(\xi - 1)$$

$$N_2^e(\xi) = \frac{1}{2}(\xi)(\xi + 1)$$

$$N_3^e(\xi) = 1 - \xi^2$$

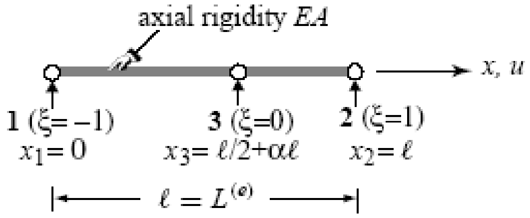

Figure 2: Isoparametric 3-node bar element.

The isoparametric definition of the 3-node straight bar element is:

$$\begin{bmatrix} 1 \\ x \\ u \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ N_3^e \end{bmatrix}$$

Where $x_1 = 0$,$x_2 = l$ and $x_3 = \left(\frac{1}{2}+\alpha\right) l$, being $-\frac{1}{2} < \alpha < \frac{1}{2}$. The Jacobian can be obtained as:

$$J = \frac{dx}{d\xi} = \sum_{i=1}^{3} x_i \frac{dN_i^e}{d\xi} = 0 \cdot \left(\xi - \frac{1}{2}\right) + l\left(\xi + \frac{1}{2}\right) + \left(\frac{1}{2}+\alpha\right) l\left(-2\xi\right) = l\left(\frac{1-4\xi\alpha}{2}\right)$$

Thus:

$$\text{For } \begin{cases} -\frac{1}{4} < \alpha < \frac{1}{4} \\ -1 \leq \xi \leq 1 \end{cases} \rightarrow 1 - 4\xi\alpha > 0 \rightarrow J > 0$$

Thus, for $\|\alpha\| \geq \frac{1}{4}$ the Jacobian will be ≤ 0. For the case where $\alpha = \frac{1}{4}$, the strain at $\xi = 1$ shows a singularity:

$$e\left(\xi\right) = \frac{du}{dx} = \sum_{i=1}^{3} u_i \frac{dN}{d\xi}\frac{d\xi}{dx} = \sum_{i=1}^{3} u_i \frac{dN}{d\xi}\frac{1}{J} = \frac{1}{l\left(\frac{1-4\xi\alpha}{2}\right)} \sum_{i=1}^{3} u_i \frac{dN}{d\xi}$$

$$e\left(1\right) = \frac{1}{l\left(\frac{1-4\frac{1}{4}}{2}\right)} \sum_{i=1}^{3} u_i \frac{dN}{d\xi} = \frac{1}{0} \sum_{i=1}^{3} u_i \frac{dN}{d\xi} \rightarrow \infty$$

Assignment 6.3

The shape functions for the 9-node plane stress element are:

$$N_1\left(\xi,\eta\right) = \frac{1}{4}\left(1-\xi\right)\left(1-\eta\right)\xi\eta \rightarrow \begin{cases} \frac{\partial N_1}{\partial \xi} &= \frac{1}{4}\left(1-2\xi\right)\left(1-\eta\right)\eta \\ \frac{\partial N_1}{\partial \eta} &= \frac{1}{4}\left(1-\xi\right)\left(1-2\eta\right)\xi \end{cases}$$

$$N_2\left(\xi,\eta\right) = -\frac{1}{4}\left(1+\xi\right)\left(1-\eta\right)\xi\eta \rightarrow \begin{cases} \frac{\partial N_2}{\partial \xi} &= -\frac{1}{4}\left(1+2\xi\right)\left(1-\eta\right)\eta \\ \frac{\partial N_2}{\partial \eta} &= -\frac{1}{4}\left(1+\xi\right)\left(1-2\eta\right)\xi \end{cases}$$

$$N_3\left(\xi,\eta\right) = \frac{1}{4}\left(1+\xi\right)\left(1+\eta\right)\xi\eta \rightarrow \begin{cases} \frac{\partial N_3}{\partial \xi} &= \frac{1}{4}\left(1+2\xi\right)\left(1+\eta\right)\eta \\ \frac{\partial N_3}{\partial \eta} &= \frac{1}{4}\left(1+\xi\right)\left(1+2\eta\right)\xi \end{cases}$$

$$N_4\left(\xi,\eta\right) = -\frac{1}{4}\left(1-\xi\right)\left(1+\eta\right)\xi\eta \rightarrow \begin{cases} \frac{\partial N_4}{\partial \xi} &= -\frac{1}{4}\left(1-2\xi\right)\left(1+\eta\right)\eta \\ \frac{\partial N_4}{\partial \eta} &= -\frac{1}{4}\left(1-\xi\right)\left(1+2\eta\right)\xi \end{cases}$$

$$N_5\left(\xi,\eta\right) = -\frac{1}{2}\left(1-\xi^2\right)\left(1-\eta\right)\eta \rightarrow \begin{cases} \frac{\partial N_5}{\partial \xi} &= \xi\left(1-\eta\right)\eta \\ \frac{\partial N_5}{\partial \eta} &= -\frac{1}{2}\left(1-\xi^2\right)\left(1-2\eta\right) \end{cases}$$

$$N_6\left(\xi,\eta\right) = \frac{1}{2}\left(1+\xi\right)\left(1-\eta^2\right)\xi \rightarrow \begin{cases} \frac{\partial N_6}{\partial \xi} &= \frac{1}{2}\left(1+2\xi\right)\left(1-\eta^2\right) \\ \frac{\partial N_6}{\partial \eta} &= -\eta\left(1+\xi\right)\xi \end{cases}$$

$$N_7\left(\xi,\eta\right) = \frac{1}{2}\left(1-\xi^2\right)\left(1+\eta\right)\eta \rightarrow \begin{cases} \frac{\partial N_7}{\partial \xi} &= -\xi\left(1+\eta\right)\eta \\ \frac{\partial N_7}{\partial \eta} &= \frac{1}{2}\left(1-\xi^2\right)\left(1+2\eta\right) \end{cases}$$

$$N_8\left(\xi,\eta\right) = -\frac{1}{2}\left(1-\xi\right)\left(1-\eta^2\right)\xi \rightarrow \begin{cases} \frac{\partial N_8}{\partial \xi} &= -\frac{1}{2}\left(1-2\xi\right)\left(1-\eta^2\right) \\ \frac{\partial N_8}{\partial \eta} &= \eta\left(1-\xi\right)\xi \end{cases}$$

$$N_9\left(\xi,\eta\right) = \left(1-\xi^2\right)\left(1-\eta^2\right) \rightarrow \begin{cases} \frac{\partial N_9}{\partial \xi} &= -2\xi\left(1-\eta^2\right) \\ \frac{\partial N_9}{\partial \eta} &= -2\eta\left(1-\xi^2\right) \end{cases}$$

The isoparametric definition of the 9-node quadrilateral element is:

$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 \\ u_{x1} & u_{x2} & u_{x3} & u_{x4} & u_{x5} & u_{x6} & u_{x7} & u_{x8} & u_{x9} \\ u_{y1} & u_{y2} & u_{y3} & u_{y4} & u_{y5} & u_{y6} & u_{y7} & u_{y8} & u_{y9} \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ N_3^e \\ N_4^e \\ N_5^e \\ N_6^e \\ N_7^e \\ N_8^e \\ N_9^e \end{bmatrix}$$

The Jacobian will be:

$$\boldsymbol{J} = \begin{bmatrix} \sum_{i=1}^{9} x_i \frac{\partial N_i}{\partial \xi} & \sum_{i=1}^{9} y_i \frac{\partial N_i}{\partial \xi} \\ \sum_{i=1}^{9} x_i \frac{\partial N_i}{\partial \eta} & \sum_{i=1}^{9} y_i \frac{\partial N_i}{\partial \eta} \end{bmatrix}$$

Assuming that the element is initially a perfect square with size l, nodes 6,7 and 8 are at the midpoint of the sides, node 9 is placed at the centre of the square, node 5 is horizontally displaced a distance αl and using a coordinates system with x=0 and y=0 placed at node 9:

$$\boldsymbol{J} = \begin{bmatrix} \frac{l}{2}\left[2\alpha\xi\left(\eta-\eta^2\right)+1\right] & 0 \\ -\frac{\alpha l}{4}\left(1-\xi^2\right)\left(1-2\eta\right) & \frac{l}{2} \end{bmatrix}$$

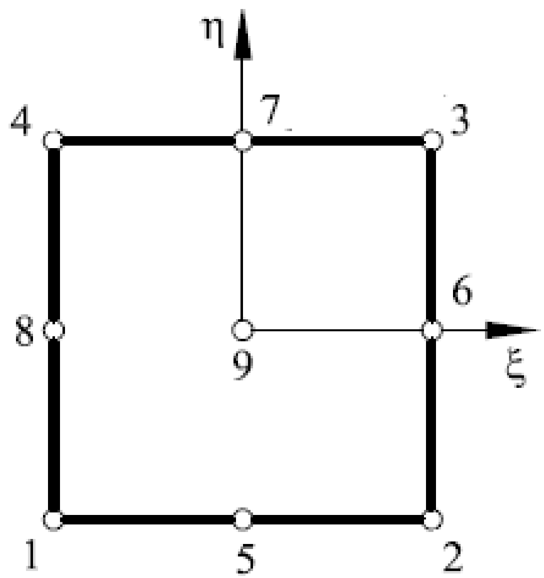

Figure 3

Node	x	y	Node	x	y
1	$-\frac{l}{2}$	$-\frac{l}{2}$	2	$\frac{l}{2}$	$-\frac{l}{2}$
3	$\frac{l}{2}$	$\frac{l}{2}$	4	$-\frac{l}{2}$	$\frac{l}{2}$
5	αl	$-\frac{l}{2}$	6	$\frac{l}{2}$	0
7	0	$\frac{l}{2}$	8	$-\frac{l}{2}$	0
9	0	0			

The determinant of the Jacobian matrix will be:

$$J(\xi,\eta) = \frac{l^2}{4}\left[2\alpha\xi\left(\eta-\eta^2\right)+1\right]$$

At node 2 ($\xi = 1$, $\eta = -1$) :

$$J(1,-1) = \frac{l^2}{4}\left[-4\alpha+1\right]$$

The evolution of the Jacobian at node 2 is depicted in figure 4. As is shown in the graph, the determinant of the Jacobian will be 0 for $\alpha = \frac{1}{4}$ and will be negative for larger values of α. This means that the determinant of the Jacobian will vanish at node 2 if node 5 moves horizontally a distance $\frac{l}{4}$. The determinant becomes negative for further displacement of this node.

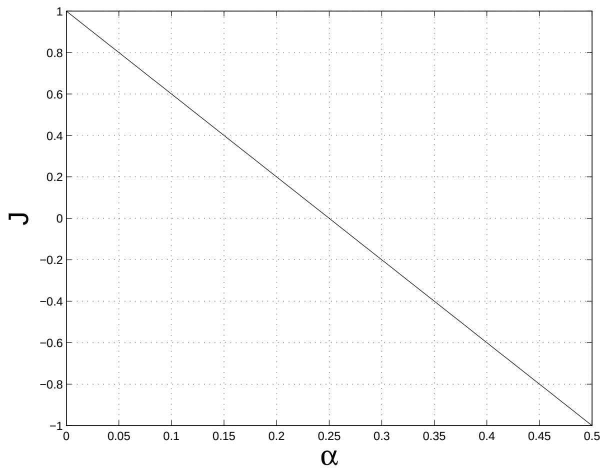

Figure 4

Assignment 6.4

The minimum integration rules of Gauss-product types that gives a rank sufficient stiffness matrix for the following elements are:

8-node hexahedron: In this case: $n = 8$, $n_F = 24$, $n_F - 6 = 18$, $minn_G = 3 \rightarrow 3$ Gauss points are needed, so use a 2x2x2 rule.

20-node hexahedron: In this case: $n = 20$, $n_F = 60$, $n_F - 6 = 52$, $minn_G = 9 \rightarrow 9$ Gauss points are needed, so use a 3x3x3 rule.

27-node hexahedron: In this case: $n = 27$, $n_F = 81$, $n_F - 6 = 75$, $minn_G = 13 \rightarrow 13$ Gauss points are needed, so use a 3x3x3 rule.

64-node hexahedron: In this case: $n = 64$, $n_F = 192$, $n_F - 6 = 186$, $minn_G = 31 \rightarrow$ 31 Gauss points are needed, so use a 4x4x4 rule.