Polytechnic University of Catalonia

Msc. in Numerical Methods

Computational Structural Mechanics and Dynamics
Assignment 1.1: "Direct Stiffness Method"

Lisandro Roldan

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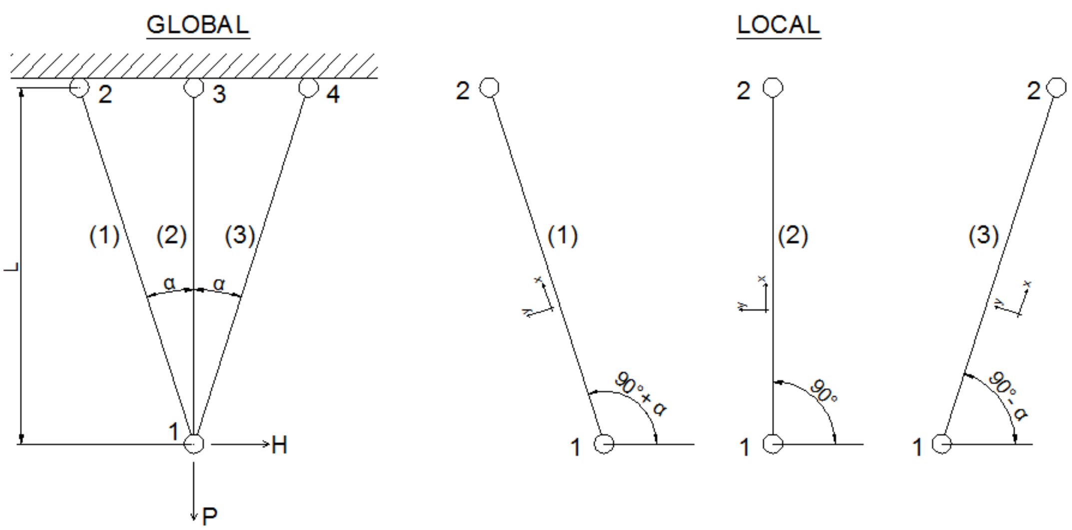
Question 1



$$A = A_1 = A_2 = A_3$$
$$E = E_1 = E_2 = E_3$$
$$L = L_2 = cL_1 = cL_2$$

Congruential transformation of element stiffness matrices

$$K^e = \frac{E^e A^e}{L^e} \begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix}$$

For element (1):

$$c \rightarrow cos(90 + \alpha) = -sin(\alpha) = -s$$
$$s \rightarrow sin(90 + \alpha) = cos(\alpha) = c$$

For element (2):

$$c \to cos(90) = 0$$
$$s \to sin(90) = 1$$

For element (3):

$$c \to cos(90 - \alpha) = sin(\alpha) = s$$
$$s \to sin(90 - \alpha) = cos(\alpha) = c$$

Plugging those trigonometrical equivalences and the length of the elements:

$$K_1 = \frac{EA}{L} \begin{bmatrix} cs^2 & -sc^2 & -cs^2 & -sc^2 \\ -sc^2 & c^3 & sc^2 & -c^3 \\ -cs^2 & sc^2 & cs^2 & -sc^2 \\ sc^2 & -c^3 & -sc^2 & c^3 \end{bmatrix}$$

$$K_2 = \frac{EA}{L} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$K_3 = \frac{EA}{L} \begin{bmatrix} cs^2 & sc^2 & -cs^2 & -sc^2 \\ sc^2 & c^3 & -sc^2 & -c^3 \\ -cs^2 & -sc^2 & cs^2 & sc^2 \\ -sc^2 & -c^3 & sc^2 & c^3 \end{bmatrix}$$

Assembling the global stiffness matrix (all of the following terms are multiplied by $\frac{EA}{L}$)

$$K_{11} = K_{11}^1 + K_{11}^2 + K_{11}^3 = cs^2 + 0 + cs^2 = 2cs^2 \qquad K_{12} = K_{12}^1 + K_{12}^2 + K_{12}^3 = -sc^2 + 0 + sc^2 = 0$$
$$K_{13} = K_{13}^1 = -cs^2 \qquad K_{14} = K_{14}^1 = sc^2$$
$$K_{15} = K_{13}^2 = 0 \qquad K_{16} = K_{14}^2 = 0$$
$$K_{17} = K_{13}^3 = -cs^2 \qquad K_{18} = K_{14}^3 = -sc^2$$
$$K_{22} = K_{22}^1 + K_{22}^2 + K_{22}^3 = c^3 + 1 + c^3 = 2c^3 + 1 \qquad K_{23} = K_{23}^1 = sc^2$$
$$K_{24} = K_{24}^1 = -c^3 \qquad K_{25} = K_{23}^2 = 0$$
$$K_{26} = K_{24}^2 = -1 \qquad K_{27} = K_{23}^3 = -sc^2$$
$$K_{28} = K_{24}^3 = -c^3 \qquad K_{33} = K_{33}^1 = cs^2$$
$$K_{34} = K_{34}^1 = -sc^2 \qquad K_{35} = 0$$
$$K_{36} = 0 \qquad K_{37} = 0$$
$$K_{38} = 0 \qquad K_{44} = K_{44}^1 = c^3$$
$$K_{45} = 0 \qquad K_{46} = 0$$
$$K_{47} = 0 \qquad K_{48} = 0$$
$$K_{55} = K_{33}^2 = 0 \qquad K_{56} = K_{34}^2 = 0$$
$$K_{57} = 0 \qquad K_{58} = 0$$
$$K_{66} = K_{44}^2 = 1 \qquad K_{67} = 0$$
$$K_{68} = 0 \qquad K_{77} = K_{33}^3 = cs^2$$
$$K_{78} = K_{34}^3 = sc^2 \qquad K_{88} = K_{44}^3 = c^3$$

The stiffness matrix takes the form of:

$$K = \frac{EA}{L} \begin{bmatrix} 2cs^2 & 0 & -cs^2 & c^2s & 0 & 0 & -cs^2 & -c^2s \\ & 1 + 2c^3 & c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \\ & & cs^2 & -c^2s & 0 & 0 & 0 & 0 \\ & & & c^3 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 \\ & & & & & & cs^2 & c^2s \\ symm & & & & & & & c^3 \end{bmatrix}$$

Forces are applied only in node (1) therefore:

$$F = [H, -P, 0, 0, 0, 0, 0, 0]'$$

The stiffness system is:

$$\frac{EA}{L}\begin{bmatrix} 2cs^2 & 0 & -cs^2 & c^2s & 0 & 0 & -cs^2 & -c^2s \\ & 1+2c^3 & c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \\ & & cs^2 & -c^2s & 0 & 0 & 0 & 0 \\ & & & c^3 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 \\ & & & & & & cs^2 & c^2s \\ symm & & & & & & & c^3 \end{bmatrix}\begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix} = \begin{bmatrix} H \\ -P \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The 5th row and column are the stiffness related to the coordinate $x3$. The only contribution to that stiffness comes from bar (2), which is placed vertically. As the bar elements can only have axial forces, its stiffness with respect to the perpendicular direction of it's axis is zero.

Question 2

By applying the boundary conditions, displacements in nodes 2,3 and 4 are restricted, leaving only two degrees of freedom. Reducing the system

$$\frac{EA}{L}\begin{bmatrix} 2cs^2 & 0 \\ 0 & 1+2c^3 \end{bmatrix}\begin{bmatrix} u_{x1} \\ u_{y1} \end{bmatrix} = \begin{bmatrix} H \\ -P \end{bmatrix}$$

Question 3

Solving the system we get:

$$u_{x1} = \frac{L}{EA}\frac{H}{2cs^2}$$
$$u_{y1} = -\frac{L}{EA}\frac{P}{1+2c^3}$$

When $\alpha \to \infty$ all the bars are vertical.
$u_{x1} \to \infty$, because there is no lateral stiffness. In other words, the system becomes hipostatic.
$u_{y1} \to \frac{-P}{3EA}$, because all the bars take the load axially in parallel.

When $\alpha \to \pi/2$ bars (1) and (3) are horizontal and have infinite length.
$u_{x1} \to \infty$, because the only contribution to the lateral stiffness comes from bar (1) and (2). The stiffness of a bar is inversely proportional to it's length ($K = EA/L$) therefore the stiffness of those bars are null. To keep the relationship $F = K.\delta$, the displacement has to go to infinite when the stiffness goes to zero.
$u_{y1} \to \frac{-PL}{EA}$, because the load is only taken by bar (2).

Question 4

To find the axial force acting in each bar it is necessary to recover first the displacements in the local coordinate system, then the elongation or contraction of each element and finally applying $F = K * \delta$ compute the force for each element.

For element (1):

$$\overline{u_{x1}^1} = -s.u_{x1} + c.u_{y1} = \frac{L}{EA}\left(-\frac{Hs}{2cs^2} - \frac{Pc}{1+2c^3}\right)$$
$$\overline{u_{x2}^1} = 0$$
$$\delta^1 = \overline{u_{x2}^1} - \overline{u_{x1}^1} = \frac{L}{EA}\left(\frac{Hs}{2cs^2} + \frac{Pc}{1+2c^3}\right)$$
$$F^1 = \frac{EA}{L/c}\frac{L}{EA}\left(\frac{Hs}{2cs^2} + \frac{Pc}{1+2c^3}\right)$$
$$F^3 = \frac{H}{2s} + \frac{Pc^2}{1+2c^3}$$

For element (2):

$$\overline{u_{x1}^2} = u_{y1} = -\frac{L}{EA}\frac{P}{1+2c^3}$$
$$\overline{u_{x2}^2} = 0$$
$$\delta^2 = \overline{u_{x2}^2} - \overline{u_{x1}^2} = \frac{L}{EA}\frac{P}{1+2c^3}$$
$$F^2 = \frac{EA}{L}\frac{L}{EA}\frac{P}{1+2c^3}$$
$$F^2 = \frac{P}{1+2c^3}$$

For element (3):

$$\overline{u_{x1}^3} = s.u_{x1} + c.u_{y1} = \frac{L}{EA}\left(\frac{Hs}{2cs^2} - \frac{Pc}{1+2c^3}\right)$$
$$\overline{u_{x2}^3} = 0$$
$$\delta^3 = \overline{u_{x2}^3} - \overline{u_{x1}^3} = \frac{L}{EA}\left(\frac{Pc}{1+2c^3} - \frac{Hs}{2cs^2}\right)$$
$$F^3 = \frac{EA}{L/c}\frac{L}{EA}\left(\frac{Pc}{1+2c^3} - \frac{Hs}{2cs^2}\right)$$
$$F^3 = \frac{Pc^2}{1+2c^3} - \frac{H}{2s}$$

When α is very small, the lateral stiffness is very low as well. To keep the relation $u_x = F_x/K_x$, the force has to tend to infinite when the stiffness tends to zero. This can also be observed when $c = 0$ is plugged into F^1 and F^3.