ASSIGNMENT 3
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ASSIGNMENT 3.1

a) We want E^* in terms of E, and ν^* in terms of ν, where E^* is a fictitious modulus in the plane stress relation and E in the plane strain.

Also ν^* is a Fictitious Poisson's ratio in the plane stress relation and ν in the plane strain relation.

So, substituting E by E^* and ν by ν^* in the plane stress relation we get:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E^*}{1-\nu^{*2}} \begin{bmatrix} 1 & \nu^* & 0 \\ \nu^* & 1 & 0 \\ 0 & 0 & \frac{1-\nu^*}{2} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{xy} \\ 2e_{xy} \end{bmatrix}$$

Comparing elements 11 of each expressions and 12 we have:

$$\left. \begin{aligned} \frac{E^*}{1-\nu^{*2}} &= \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \\ \frac{E^*}{(1-\nu^{*2})}\nu^* &= \frac{E\cancel{(1-\nu)}\,\nu}{(1+\nu)(1-2\nu)\cancel{(1-\nu)}} \end{aligned} \right\}$$

$$\frac{\cancel{E}(1-\nu)}{\cancel{(1+\nu)}\cancel{(1-2\nu)}}\,\nu^* = \frac{\cancel{E}\,\nu}{\cancel{(1+\nu)}\cancel{(1-2\nu)}}$$

$$\boxed{\nu^* = \frac{\nu}{1-\nu}}$$

$$\frac{E^*}{(1-\nu^{*2})} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$$

$$\Big\downarrow\ \nu^* = \frac{\nu}{1-\nu}$$

$$\frac{E^*}{1-\left(\frac{\nu}{1-\nu}\right)^2} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$$

$$\frac{E^*}{\frac{1+\cancel{\nu^2}-2\nu-\cancel{\nu^2}}{(1-\nu)^2}} = \frac{E^*}{\frac{1-2\nu}{(1-\nu)^2}} = E^*\frac{(1-\nu)^{\cancel{2}}}{\cancel{(1-2\nu)}} = \frac{E\,\cancel{(1-\nu)}}{(1+\nu)\cancel{(1-2\nu)}}$$

$$E^* = \frac{E}{(1+\nu)(1-\nu)}$$

$$\boxed{E^* = \frac{E}{1-\nu^2}}$$

We can do the inverse process: go from plane strain to plain stress by replacing a fictitious modulus and Poisson's ratio in the plane strain constitutive matrix. Let's do it:

We call E^* to the fictitious modulus and ν^* to the Poisson's ratio.

And we do it again with the elements 11 and 12:

$$\left\{ \begin{array}{l} \dfrac{E^*(1-\nu^*)}{(1+\nu^*)(1-2\nu^*)} = \dfrac{E}{1-\nu^2} \\[2ex] \dfrac{E^*(1-\nu^*)}{(1+\nu^*)(1-2\nu^*)} \dfrac{\nu^*}{1-\nu^*} = \dfrac{E\nu}{1-\nu^2} \end{array} \right.$$

$$\downarrow$$

$$\frac{\cancel{E}}{1-\nu^2} \frac{\nu^*}{1-\nu^*} = \frac{\cancel{E}\nu}{1-\nu^2}$$

$$\frac{\nu^*}{1-\nu^*} = \frac{\nu}{\cancel{(1-\nu^2)}} \cancel{(1-\nu^2)}$$

$$\nu^* = \nu(1-\nu^*) = \nu - \nu\nu^*$$

$$\nu^*(1+\nu) = \nu$$

$$\boxed{\nu^* = \frac{\nu}{1+\nu}}$$

And:

$$\frac{E^*(1-\nu^*)}{(1+\nu^*)(1-2\nu^*)} = \frac{E}{1-\nu^2}$$

$$\nu^* = \frac{\nu}{1+\nu}$$

$$\frac{E^*\left(1-\frac{\nu}{1+\nu}\right)}{\left(1+\frac{\nu}{1+\nu}\right)\left(1-2\frac{\nu}{1+\nu}\right)} = \frac{E^*\frac{1}{1+\nu}}{\left(\frac{1+2\nu}{1+\nu}\right)\left(\frac{1-\nu}{1+\nu}\right)} =$$

$$= E^* \frac{1}{\cancel{(1+\nu)}} \frac{(1+\nu)\cancel{(1+\nu)}}{(1+2\nu)(1-\nu)} = \frac{E}{(1+\nu)(1-\nu)}$$

$$E^* = \frac{E}{(1+\nu)\cancel{(1-\nu)}} \frac{(1+2\nu)\cancel{(1-\nu)}}{(1+\nu)} = \frac{1+2\nu}{(1+\nu)^2}E$$

$$\boxed{E^* = \frac{1+2\nu}{(1+\nu)^2}E}$$

b)

$$U = \frac{1}{2}\left[\sigma_{xx} e_{xx} + \sigma_{yy} e_{yy} + \sigma_{xy} e_{xy} + \sigma_{yx} e_{yx}\right] =$$

$$= \frac{1}{2}\left[\sigma_{xx} e_{xx} + \sigma_{yy} e_{yy} + 2\sigma_{xy} e_{xy}\right] =$$

$$= \frac{1}{2}\begin{pmatrix} e_{xx} & e_{yy} & 2e_{xy} \end{pmatrix} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix} = \frac{1}{2} e^T \sigma = (*)$$

In terms of strains:

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix} = \underbrace{\frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{pmatrix} 1 & \frac{\nu}{1-\nu} & 0 \\ \frac{\nu}{1-\nu} & 1 & 0 \\ 0 & 0 & \frac{1-2\nu}{2(1-\nu)} \end{pmatrix}}_{E} \underbrace{\begin{pmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{pmatrix}}_{e} =$$

$$(*) = \frac{1}{2} e^T E e$$

$$\boxed{U = \frac{1}{2} e^T E e}$$

the internal energy can be written in terms of stresses:

$U = \frac{1}{2}\sigma^T C \sigma$, where $C = E^{-1}$

$$U = \frac{1}{2}\left(\sigma_{xx}e_{xx} + \sigma_{yy}e_{yy} + \sigma_{xy}e_{xy} + \sigma_{yx}e_{yx}\right] =$$

$$= \frac{1}{2}\left(\sigma_{xx}e_{xx} + \sigma_{yy}e_{yy} + 2\sigma_{xy}e_{xy}\right) =$$

$$= \frac{1}{2}\left(\sigma_{xx}\ \sigma_{yy}\ \sigma_{xy}\right)\begin{pmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{pmatrix} = \frac{1}{2}\sigma^T\begin{pmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{pmatrix} =$$

$$= \frac{1}{2}\sigma^T \underbrace{E^{-1}E}_{I}\begin{pmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{pmatrix} = \frac{1}{2}\sigma^T E^{-1}\underbrace{E\begin{pmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{pmatrix}}$$

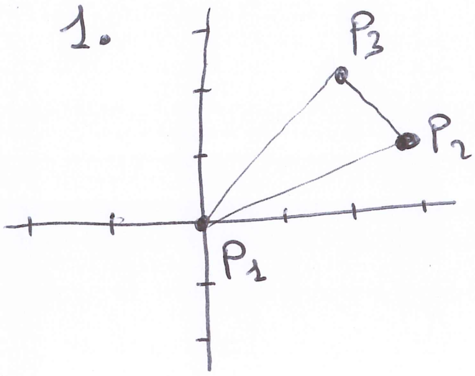
$$E = \begin{pmatrix} \frac{1}{1-\nu^2} & \frac{\nu}{1-\nu^2} & 0 \\ \frac{\nu}{1-\nu^2} & \frac{1}{1-\nu^2} & 0 \\ 0 & 0 & \frac{1-\nu}{2(1-\nu^2)} \end{pmatrix}$$ in plane stress expression

$$\frac{E}{1-\nu^2}\begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{pmatrix}\begin{pmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{pmatrix} = \overbrace{\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix}} = \sigma$$

$$\boxed{U = \frac{1}{2}\sigma^T \underbrace{E^{-1}}_{C}\underbrace{Ee}_{\sigma} = \frac{1}{2}\sigma^T C \sigma}$$

ASSIGNMENT 3.2.

1.



$P_1(0,0)$ $\quad P_1P_2 = \sqrt{10}$

$P_2(3,1)$ $\quad P_1P_3 = \sqrt{8}$

$P_3(2,2)$ $\quad P_2P_3 = \sqrt{2}$

$h = 1$, we suppose h is constant. So we must use:

$$K^e = \frac{h}{4A} \begin{pmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{pmatrix} \begin{pmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{pmatrix} \begin{pmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{pmatrix}$$

A is the ~~t~~triangle area. We compute it by Heron Formula.

$$s = \text{semiperimeter} = \frac{1}{2}(P_1P_2 + P_1P_3 + P_2P_3) = \frac{\sqrt{2}}{2}(\sqrt{5} + 3)$$

$$A = \sqrt{s(s - P_1P_2)(s - P_1P_3)(s - P_2P_3)}$$

$$A \simeq 2$$

Taking into account that:

$$Y_{jk} = y_j - y_k \qquad x_{jk} = x_j - x_k$$

$$\left\{\begin{array}{l} Y_{23} = Y_2 - Y_3 = -1 \\ Y_{31} = Y_3 - Y_1 = 2 \\ Y_{12} = Y_1 - Y_2 = -1 \end{array}\right\} \quad \begin{array}{l} X_{21} = 3 \\ x_{32} = -1 \\ x_{13} = -2 \end{array}$$

$$K^e = \frac{1}{8} \times 25 \begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 2 & 0 & -2 \\ 0 & -2 & 2 \\ -1 & 0 & 3 \\ 0 & 3 & -1 \end{pmatrix} \begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 3 \\ -1 & -1 & -2 & 2 & 3 & -1 \end{pmatrix} =$$

$$= \frac{25}{8} \begin{pmatrix} 6 & 3 & -4 & -2 & -2 & -1 \\ 3 & 6 & 2 & 4 & -5 & 10 \\ -4 & 2 & 24 & -12 & -20 & 10 \\ -2 & 4 & -20 & 14 & 14 & -28 \\ -2 & -5 & 14 & 22 & 22 & -9 \\ -1 & -10 & -28 & -28 & -9 & 38 \end{pmatrix}$$

So :

$$K_e = \begin{pmatrix} 18.7500 & 9.3750 & -12.5000 & -6.2500 & -6.2500 & -3.1250 \\ 9.3750 & 18.7500 & 6.2500 & 12.5000 & -15.6250 & -31.2500 \\ -12.5000 & 6.2500 & 75.0000 & -37.5000 & -62.5000 & 31.2500 \\ -6.2500 & 12.5000 & -37.5000 & 75.0000 & 43.7500 & -87.5000 \\ -6.2500 & -15.6250 & -62.5000 & 43.7500 & 68.7500 & -28.1250 \\ -3.1250 & -31.2500 & 31.2500 & -87.5000 & -28.1250 & 118.7500 \end{pmatrix}$$

So $k_{11} = 18{,}75$ $\quad k_{21} = 9{,}375 \ \ldots$

$k_{21} = 9{,}375$

$\vdots$

$\ldots \quad k_{66} = 118{,}75$

2. $(r_1 + r_3 + r_5) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

r_i = row i

c_i = column i

$r_2 + r_4 + r_6 = 0$

$c_1 + c_3 + c_5 = 0$

$c_2 + c_4 + c_6 = 0$

$\sum r_i = \sum c_i = 0$

because K_e is symmetric

$K_e u = F$

$$\begin{pmatrix} F_{x_1} \\ F_{y_1} \\ F_{x_2} \\ F_{y_2} \\ F_{x_3} \\ F_{y_3} \end{pmatrix} = K_e \begin{pmatrix} u_{x_1} \\ u_{y_1} \\ u_{x_2} \\ u_{y_2} \\ u_{x_3} \\ u_{y_3} \end{pmatrix}$$

$F_{x_1} = k_{11} u_{x_1} + \dots + k_{16} u_{y_3}$

$F_{y_1} = k_{21} u_{x_1} + \quad + k_{26} u_{y_3}$

$F_{x_2} = k_{31} u_{x_1} + \dots$

$\vdots$

$F_{y_3} = \dots$

columns

$k_{11}+k_{13}+k_{15}=0$

$k_{21}+k_{23}+k_{25}=0$

$k_{31}+k_{33}+k_{35}=0$

$k_{41}+k_{43}+k_{45}=0$

$k_{51}+k_{53}+k_{55}=0$

$k_{61}+k_{63}+k_{65}=0$

$\rightarrow k_{12}+k_{14}+k_{16}=0$

$k_{22}+k_{44}+k_{66}=0$

$k_{32}+k_{34}+k_{36}=0$

$k_{42}+k_{44}+k_{46}=0$

$k_{52}+k_{54}+k_{56}=0$

$k_{62}+k_{64}+k_{66}=0$

rows

$\rightarrow k_{11}+k_{31}+k_{51}=0$

$\rightarrow k_{13}+k_{33}+k_{53}=0$

$\rightarrow k_{15}+k_{35}+k_{55}=0$

$k_{12}+k_{32}+k_{52}=0$

$\rightarrow k_{14}+k_{34}+k_{54}=0$

$\rightarrow k_{16}+k_{36}+k_{56}=0$

$k_{21}+k_{41}+k_{61}=0$

$k_{22}+k_{44}+k_{66}=0$

$k_{23}+k_{43}+k_{63}=0$

$k_{24}+k_{44}+k_{64}=0$

$k_{25}+k_{45}+k_{65}=0$

$k_{26}+k_{46}+k_{66}=0$

Sum of Forces on X coordinate

$$f_{x_1}+f_{x_3}+f_{x_5}=k_{11}u_{x_1}+k_{12}u_{y_1}+k_{13}u_{x_2}+k_{14}u_{y_2}+k_{15}u_{x_3}+$$
$$+k_{16}u_{y_3}+k_{31}u_{x_1}+k_{32}u_{y_1}+k_{33}u_{x_2}+k_{34}u_{y_2}+k_{35}u_{x_3}+$$
$$+k_{36}u_{y_3}+k_{51}u_{x_1}+k_{52}u_{y_1}+k_{53}u_{x_2}+k_{54}u_{y_2}+k_{55}u_{x_3}+$$
$$+k_{56}u_{y_3}=\underbrace{(k_{11}+k_{31}+k_{51})}_{=0}u_{x_1}+\underbrace{(k_{12}+k_{14}+k_{16})}_{=0}u_{y_1}$$
$$+\underbrace{(k_{13}+k_{33}+k_{53})}_{=0}u_{x_2}+\underbrace{(k_{14}+k_{34}+k_{54})}_{=0}u_{y_2}+$$
$$+\underbrace{(k_{15}+k_{35}+k_{55})}_{=0}u_{x_3}+\underbrace{(k_{16}+k_{36}+k_{56})}_{=0}u_{y_3}$$

We can se that $f_{x_1}+f_{x_2}+f_{x_3}=0$
(because of the → selected)
Similarly we will find that $f_{y_1}+f_{y_2}+f_{y_3}=0$

So $\sum f_{x_i}=0$
and $\sum f_{y_i}=0$

The system is in equilibrium of forces. is

the reason of $\begin{cases} \sum r_i = 0 \\ \sum c_i = 0 \end{cases}$