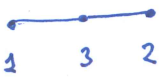
ASSIGNMENT 4
JORGE BALSA GONZÁLEZ

ASSIGNMENT 4.1



Natural coordinates for ξ nodes
$$\begin{cases} (1) & \xi = -1 \\ (3) & \xi = 0 \\ (2) & \xi = 1 \end{cases}$$

- node value conditions

node 1 e: element 1
$$\begin{cases} N_1^e = 1 & \text{for } \xi = -1 \\ N_1^e = 0 & \text{for } \xi = 0 \\ N_1^e = 0 & \text{for } \xi = 1 \end{cases}$$

node 2
$$\begin{cases} N_2^e = 0 & \text{for } \xi = -1 \\ N_2^e = 0 & \text{for } \xi = 0 \\ N_2^e = 1 & \text{for } \xi = 1 \end{cases}$$

node 3
$$\begin{cases} N_3^e = 0 & \text{for } \xi = -1 \\ N_3^e = 1 & \text{for } \xi = 0 \\ N_3^e = 0 & \text{for } \xi = 1 \end{cases}$$

$$\begin{cases} N_1^e = a_0 + a_1 \xi + a_2 \xi^2 \\ N_1^e = -\frac{1}{2}\xi + \frac{1}{2}\xi^2 \\ N_1^e = \frac{1}{2}\xi(\xi - 1) \end{cases} \qquad \begin{cases} a_0 = 0 \\ a_1 = -\frac{1}{2} \\ a_2 = \frac{1}{2} \end{cases}$$

$$N_2^{(e)}(\xi) = b_0 + b_1 \xi + b_2 \xi^2$$
$$N_2^{(e)}(\xi) = \frac{1}{2}\xi + \frac{1}{2}\xi^2 \qquad \begin{cases} b_0 = 0 \\ b_1 = \frac{1}{2} \\ b_2 = \frac{1}{2} \end{cases}$$
$$N_2^{(e)}(\xi) = \frac{1}{2}\xi(\xi + 1)$$

$$N_3^{(e)} = c_0 + c_1\xi + c_2\xi^2 \qquad \left\{ \begin{array}{l} c_0 = 1 \\ c_1 = 0 \\ c_2 = -1 \end{array} \right.$$

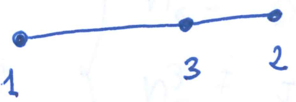
$$N_3^{(e)} = 1-\xi^2 = (1+\xi)(1-\xi)$$

- We can see:

$$N_1^{(e)} + N_2^{(e)} + N_3^{(e)} = \frac{1}{2}\xi^2 - \cancel{\frac{1}{2}\xi} + \frac{1}{2}\xi^2 + \cancel{\frac{1}{2}\xi} + 1 - \xi^2 = \cancel{\xi^2} - \cancel{\xi^2} + 1$$

$$\boxed{N_1^{(e)} + N_2^{(e)} + N_3^{(e)} = 1}$$

ASSIGNMENT 4.2.



$x_2 = \ell$

$x_1 = 0$

$x_3 = \frac{\ell}{2} + \alpha\ell$

$-\frac{1}{2} < \alpha < \frac{1}{2}$

1.a)

$$x = \underset{\overset{\shortparallel}{0}}{x_1} N_1^{(e)} + x_2 N_2^{(e)} + x_3 N_3^{(e)} = \ell N_2^{(e)} + \left(\frac{1}{2}+\alpha\right)\ell N_3^{(e)}$$

Remembering $N_2(\xi) = \frac{1}{2}\xi(\xi+1)$ when node 3 is centered ($\alpha = 0$)

So $N_2(\xi) = \left(\frac{1}{2}+\alpha\right)\xi(\xi+1)$

and similarly: $N_3(\xi) = 1-\xi^2$

So we have $x = \ell N_2 + \left(\frac{1}{2}+\alpha\right)\ell N_3$

$$x = \left(\frac{1}{2}+\alpha\right)\xi(\xi+1)\,\ell + \left(\frac{1}{2}+\alpha\right)\ell\,(1-\xi^2)$$

$$x=\left(\frac{1}{2}+\alpha\right)\ell\left(\cancel{\xi^2}+\xi-\cancel{\xi^2}+1\right)$$

$$x=\left(\frac{1}{2}+\alpha\right)\ell\,(\xi+1)$$

$$J=\frac{dx}{d\xi}=\left(\frac{1}{2}+\alpha\right)\ell$$

IF $-\frac{1}{4}<\alpha<\frac{1}{4}$ $\quad\left(\frac{1}{2}+\alpha\right)>0$, so $J>0$ always.

b) $\alpha=0$ $\quad J=\left(\frac{1}{2}+0\right)\ell=\frac{\ell}{2}$

$J=\frac{\ell}{2}$ is a constant, over the element.

2.

$$e=\frac{du}{dx}=Bu^e \qquad u^e=\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}$$

$$B=\frac{dN}{dx}=J^{-1}\frac{dN}{d\xi}$$

$$N=[N_1\ N_2\ N_3] \qquad J=\left(\frac{1}{2}+\alpha\right)\ell \qquad J^{-1}=\frac{1}{\left(\frac{1}{2}+\alpha\right)\ell}$$

We have seen (page 2): $N_2=\left(\frac{1}{2}+\alpha\right)\xi(\xi+1)$

$N_3=1-\xi^2$

also $N_1=\left(\frac{1}{2}+\alpha\right)(\xi^2-\xi)$

$$B = J^{-1}\frac{d}{d\xi}\left(N_1 \; N_2 \; N_3\right) = J^{-1}\frac{d}{d\xi}\left(\left(\tfrac{1}{2}+\alpha\right)(\xi^2-\xi),\ \left(\tfrac{1}{2}+\alpha\right)(\xi^2+\xi),\ (1-\xi^2)\right)$$

$$B = J^{-1}\left(\left(\tfrac{1}{2}+\alpha\right)(2\xi-1),\ \left(\tfrac{1}{2}+\alpha\right)(2\xi+1),\ -2\xi\right)$$

$$B = \frac{1}{\left(\frac{1}{2}+\alpha\right)\ell}\begin{pmatrix} \left(\frac{1}{2}+\alpha\right)(2\xi-1) & \left(\frac{1}{2}+\alpha\right)(2\xi+1) & -2\xi \end{pmatrix}$$

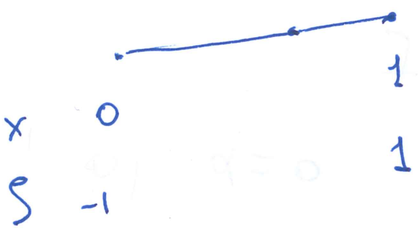
3. $K^e = \int_0^\ell EA\,B^T B\,dx$

$dx = \det J\,d\xi$

$J = \det J$

$dx = J\,d\xi$

$J = \det J = \left(\tfrac{1}{2}+\alpha\right)\ell$



$$\int_0^1 dx \longleftrightarrow \int_{-1}^{1} J\,d\xi$$

We can compute the element stiffness matrix, as we can do with a bar element.

$$U^e = \frac{1}{2}\int_0^\ell e\,EA\,e\,dx = \frac{1}{2}(u^e)^T K^e u^e$$

$$\cancel{\tfrac{1}{2}}(u^e)^T k^e u^e = \cancel{\tfrac{1}{2}}\int_0^l e\,E\,A\,e\,dx$$

$e = Bu$

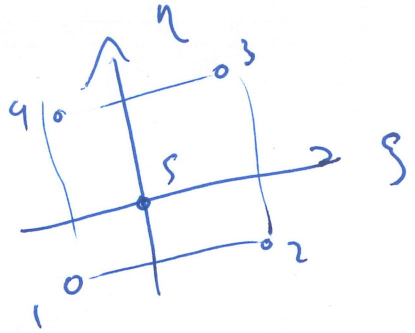
$B = eu^{-1}$

$$k^e = \int_0^l \underbrace{(u^T)^{-1} e}_{B^T}\, E A\, \underbrace{e\, u^{-1}}_{B}$$

$$k^e = \int_0^l B^T E A B\,dx = \int_0^l E A B^T B\,dx =$$

$J = \frac{dx}{d\xi}$ $\quad dx = J.d\xi \longrightarrow = \int_{-1}^{1} E A B^T B J d\xi$

ASSIGNMENT 4.3



We can suppose that we have a 9-node biquadratic Quadrilateral elements:

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta)\xi\eta$$

$$N_2 = \frac{1}{4}(1+\xi)(1-\eta)\xi\eta$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta)\xi\eta$$

$$N_4 = \frac{1}{4}(\xi-1)(\eta-1)\xi\eta$$

But we neglect this nodes

$$N_5 = \frac{1}{2}(1-\xi^2)(\eta-1)\eta$$

$$N_6 = \frac{1}{2}(\xi+1)(1-\eta^2)\xi$$

$$N_7 = \frac{1}{2}(1-\xi^2)(\eta+1)\eta$$

$$N_8 = \frac{1}{2}(\xi-1)(1-\eta^2)\xi$$

$$N_9 = (1-\xi^2)(1-\eta^2)$$

and now $N_9 = N_5$. We will find N_5 !

We now that (1) $\sum_{i=1}^{5} N_i = 1$

(2)

$$N_1 + N_2 + N_3 + N_4 + N_5 = 1$$

$$N_1 + N_2 + N_3 + N_4 + \alpha N_5 = 0$$

$$N_1 + N_2 + N_3 + N_4 + N_5 = 1$$

$$N_1 + N_2 + N_3 + N_4 = \xi\eta$$

$$(1) \quad \xi\eta + N_5 = 1$$

$$(2) \quad \xi\eta + \alpha N_5 = 0$$

$$(\alpha - 1) N_5 = -1 \qquad \boxed{\alpha = \frac{N_5 - 1}{N_5}}$$

$$N_5 = \frac{-1}{\alpha - 1}$$

$$(2) \rightarrow \quad \xi\eta + \frac{N_5 - 1}{\cancel{N_5}} \cancel{N_5} = 0$$

$$\boxed{N_5 = 1 - \xi\eta} \qquad \boxed{\alpha = \frac{1 - \xi\eta - 1}{1 - \xi\eta} = \frac{\xi\eta}{\xi\eta - 1}}$$

So (1) and (2) are satisfied when $N_5 = 1 - \xi\eta$ and

$$\alpha = \frac{\xi\eta}{\xi\eta - 1} = \frac{-\xi\eta}{N_5} = \frac{-(N_1 + N_2 + N_3 + N_4)}{N_5}$$