ASSIGNMENT 5
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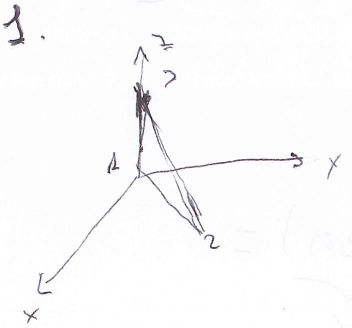
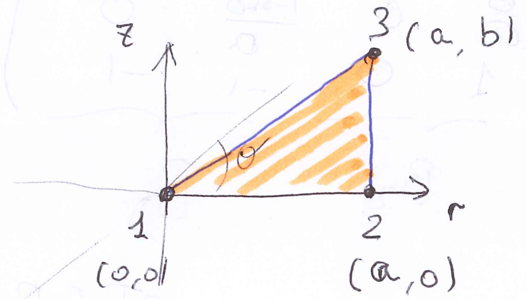
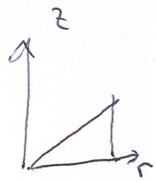
ASSIGNMENT 5.1

1.







$$\begin{cases} r_1 = 0 & z_1 = 0 \\ r_2 = a & z_2 = 0 \\ r_3 = a & z_3 = b \end{cases}$$

In these coordinates:

$$N_1 = r$$
$$N_2 = Z$$
$$N_3 = 1 - r - Z$$

$$B_i = \begin{pmatrix} q_{r_i} & 0 \\ 0 & q_{z_i} \\ q_{\theta_i} & 0 \\ q_{z_i} & q_{r_i} \end{pmatrix}$$

$$q_{r_i} = \frac{\partial N_i^{(e)}}{\partial r}$$
$$q_{z_i} = \frac{\partial N_i^{(e)}}{\partial z}$$
$$q_{\theta_i} = \frac{\partial N_i^{(e)}}{r}$$

$$B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \quad B_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ \frac{z}{r} & 0 \\ 1 & 0 \end{pmatrix} \quad B_3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ \frac{1-r-z}{r} & 0 \\ -1 & -1 \end{pmatrix}$$

$$\left(\underbrace{\begin{matrix} 1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{matrix}}_{B_1} \vdots \underbrace{\begin{matrix} 0 & 0 \\ 0 & 1 \\ \frac{b}{a} & 0 \\ 1 & 0 \end{matrix}}_{B_2} \vdots \underbrace{\begin{matrix} -1 & 0 \\ 0 & -1 \\ \frac{1-ab}{a} & 0 \\ -1 & -1 \end{matrix}}_{B_3}\right) \qquad A = \frac{ab}{2}$$

$$k^e = \int_V B^T E B \, dV = B^T E B \, (A \, 2\pi) =$$

$$= \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & b/a & 1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & \frac{1-a-b}{a} & -1 \\ 0 & -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} B \, (A 2\pi) =$$

$$= \frac{ab}{2} 2\pi \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & b/a & 1/2 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & \frac{1-a-b}{a} & -1/2 \\ 0 & -1 & 0 & -1/2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 1 & 0 & b/a & 0 & \frac{1-a-b}{a} & 0 \\ 0 & 1 & 1 & 0 & -1 & -1 \end{pmatrix} =$$

$$= \pi ab \begin{pmatrix} 2 & 0 & b/a & 0 & -1+\frac{1-a-b}{a} & 0 \\ 0 & 1/2 & 1/2 & 0 & -1/2 & -1/2 \\ b/a & 1/2 & b^2/a^2+\frac{1}{2} & 0 & \left(\frac{b}{a}\frac{1-a-b}{a}-\frac{1}{2}\right) & -1/2 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ -1+\frac{1-a-b}{a} & \left(-\frac{1}{2}\right) & \left(\frac{b}{a}\left(\frac{1-a-b}{a}\right)-\frac{1}{2}\right) & 0 & \left(1+\left(\frac{1-a-b}{a}\right)^2+\frac{1}{2}\right) & \frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} & -1 & \left(+\frac{1}{2}\right) & 1+\frac{1}{2} \end{pmatrix}$$

2. We can see that the sum of the rows 2, 4 and 6 vanish:

$0+0+0=0$

$\frac{1}{2}+0-\frac{1}{2}=0$

$\frac{1}{2}+0-\frac{1}{2}=0$

$0+1-1=0$

$-\frac{1}{2}+0+\frac{1}{2}=0$

$-\frac{1}{2}-1+1+\frac{1}{2}=0$

Also with the rows 2, 4 and 6:

$0+0+0=0$

$\frac{1}{2}+0-\frac{1}{2}=0$

$\frac{1}{2}+0-\frac{1}{2}=0$

$0+1-1=0$

$-\frac{1}{2}+0+\frac{1}{2}=0$

$-\frac{1}{2}-1+1+\frac{1}{2}=0$

But the sum of columns (or rows) 1, 3 and 5 is not zero.

$$2+\frac{b}{a}-1+\frac{1-a-b}{a}=1+\frac{1-a-b+b}{a}=1+\underbrace{\frac{1-a}{a}}_{\frac{1}{a}\text{ never can be }0\text{ for }a\in\mathbb{R}^+}\neq 0$$

As we now:

$$\begin{pmatrix} F_{r_1} \\ F_{z_1} \\ F_{r_2} \\ F_{z_2} \\ F_{r_3} \\ F_{z_3} \end{pmatrix} = K \begin{pmatrix} u_{r_1} \\ u_{\theta_1} \\ u_{r_2} \\ u_{\theta_2} \\ u_{r_3} \\ u_{\theta_3} \end{pmatrix}$$

This can be interpreted in this way:

- $\sum F_{z_i}=0$

but

- $\sum F_{r_i}\neq 0$

The system is in equilibrium of forces in the z (axis) direction (This is for its symmetry). But it is not in equilibrium of forces in the r (axis) direction. This is because the forces that appears with the ~~revo~~lution of the triangle aroun z axis. (centrifuge and centripetal forces)

3.

$$b = [0, -g]^T$$

We can think in gravity force as an external force acting on the system.

$$F_{ext} = \int_A N^T b\, r\, dA$$

$$F_i^{(e)} = \iint N_i b\, r\, dr\, dz$$

$$F_1 = \int_0^0 \int_0^0 N_1 \begin{pmatrix} 0 \\ -g \end{pmatrix} r\, dr\, dz = 0$$

$$F_2 = \int_0^0 \int_0^a N_2 \begin{pmatrix} 0 \\ -g \end{pmatrix} r\, dr\, dz = 0$$

$$F_3 = \int_0^b \int_0^a N_3 \begin{pmatrix} 0 \\ -g \end{pmatrix} r\, dr\, dz = \int_0^b \int_0^a (1 - r - z) \begin{pmatrix} 0 \\ -g \end{pmatrix} r\, dr\, dz =$$

$$F_3 = \int_0^b \left(\frac{r^2}{2} - \frac{r^3}{3} - z r \right) \Bigg|_0^a \begin{pmatrix} 0 \\ -g \end{pmatrix} dz = \int_0^b \left(\frac{a^2}{2} - \frac{a^3}{3} - a z \right) \cdot$$

$$\cdot \begin{pmatrix} 0 \\ -g \end{pmatrix} dz = \left(\frac{a^2}{2} z - \frac{a^3}{3} z - \frac{a z^2}{2} \right) \Bigg|_0^b \begin{pmatrix} 0 \\ -g \end{pmatrix} =$$

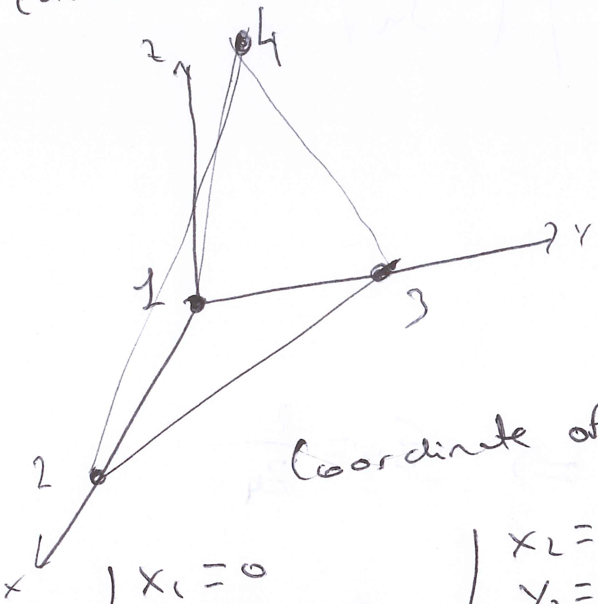
$$F_3 = \left(\frac{a^2}{2} b - \frac{a^3}{3} b - \frac{ab^2}{2} \right) \begin{pmatrix} 0 \\ -g \end{pmatrix}$$

$$F_3 = ab \left(\frac{a}{2} - \frac{a^2}{3} - \frac{b}{2} \right) \begin{pmatrix} 0 \\ -g \end{pmatrix}$$

$$F = ab \, \frac{a(3-2a) - 3b}{6} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -g \end{pmatrix}$$

ASSIGNMENT 5.2.

As 3 points defines a plane, I can draw in this convenient way:



$$V = \frac{1}{6}\begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix}$$

Coordinate of the 4 nodes:

$$\begin{cases} x_1 = 0 \\ y_1 = 0 \\ z_1 = 0 \end{cases} \quad \begin{cases} x_2 = \\ y_2 = 0 \\ z_2 = 0 \end{cases} \quad \begin{cases} x_3 = 0 \\ y_3 = \\ z_3 = 0 \end{cases} \quad \begin{cases} x_4 \\ y_4 \\ z_4 \end{cases}$$

$$V = \frac{1}{6}\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & x_2 & 0 & x_4 \\ 0 & 0 & y_3 & y_4 \\ 0 & 0 & 0 & z_4 \end{vmatrix} = \frac{1}{6}\begin{vmatrix} x_2 & 0 & x_4 \\ 0 & y_3 & y_4 \\ 0 & 0 & z_4 \end{vmatrix} = \frac{1}{6}(x_2 y_3 z_4)$$

$$\begin{pmatrix} 1 \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & x_2 & 0 & x_4 \\ 0 & 0 & y_3 & y_4 \\ 0 & 0 & 0 & z_4 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix}$$

$$1 = \xi_1 + \xi_2 + \xi_3 + \xi_4$$

$$x = x_2 \xi_2 + x_4 \xi_4$$

$$y = y_3 \xi_3 + y_4 \xi_4$$

$$z = z_4 \xi_4 \Rightarrow \xi_4 = \frac{z}{z_4}$$

$$x = x_2 \xi_2 + x_4 \xi_4 = x_2 \xi_2 + x_4 \frac{z}{z_4}$$

$$y = y_3 \xi_3 + y_4 \xi_4 = y_3 \xi_3 + y_4 \frac{z}{z_4}$$

$$x_2 \xi_2 = x - x_4 \frac{z}{z_4} = \frac{z_4 x - x_4 z}{z_4} \qquad \boxed{\xi_2 = \frac{z_4 x - x_4 z}{z_4 x_2}}$$

$$y_3 \xi_3 = y - y_4 \frac{z}{z_4} = \frac{z_4 y - y_4 z}{z_4} \qquad \boxed{\xi_3 = \frac{z_4 y - y_4 z}{z_4 y_3}}$$

$$\xi_1 = 1 - \frac{z_4 x - x_4 z}{z_4 x_2} - \frac{z_4 y - y_4 z}{z_4 y_3} - \frac{z}{z_4}$$

$$\xi_1 = \frac{z_4 x_2 y_3 - z_4 y_3 x + x_4 y_3 z - z_4 x_2 y + x_2 y_4 z - x_2 y_3 z}{z_4 x_2 y_3}$$

$$F^e_{ext} = \int_{V^e} N^T b \, dV$$

$$F^e_{ext} = \frac{1}{6}(x_2 y_3 z_4) N^T b$$

$$N = \begin{pmatrix} \xi_1 & 0 & 0 & \xi_2 & 0 & 0 & \xi_3 & 0 & 0 & \xi_4 & 0 & 0 \\ 0 & \xi_1 & 0 & 0 & \xi_2 & 0 & 0 & \xi_3 & 0 & 0 & \xi_4 & 0 \\ 0 & 0 & \xi_1 & 0 & 0 & \xi_2 & 0 & 0 & \xi_3 & 0 & 0 & \xi_4 \end{pmatrix}$$

$$\cancel{F_{ext} = \begin{pmatrix} F_{x_1} \\ F_{y_1} \\ F_{z_1} \\ F_{x_2} \\ F_{y_2} \\ F_{z_2} \\ F_{x_3} \\ F_{y_3} \\ F_{z_3} \\ F_{x_4} \\ F_{y_4} \\ F_{z_4} \end{pmatrix}}$$

There is No force in Face 4,

so $F_{x_4} = 0$

$F_{y_4} = 0$

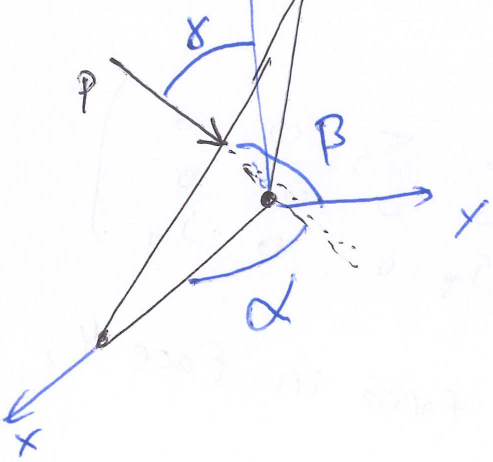
$F_{z_4} = 0$

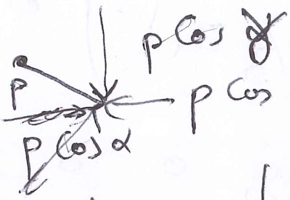
P_i presure acting normal to the face i

$P_1 = P_2 = P_3 = P$

$$F_{ex} = \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix}$$

Each face has a p acting normal to it.





$$f_1 = p A_1 \quad \begin{pmatrix} f_{1x} \\ f_{1y} \\ f_{1z} \end{pmatrix} = pA_1 \begin{pmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{pmatrix}$$

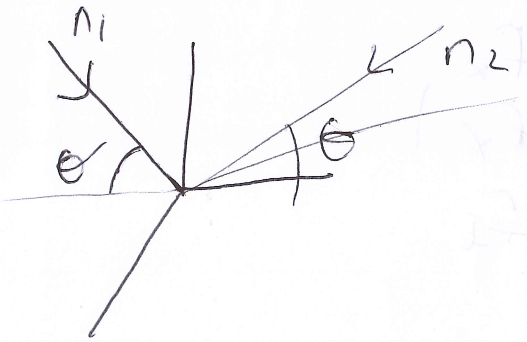
So:

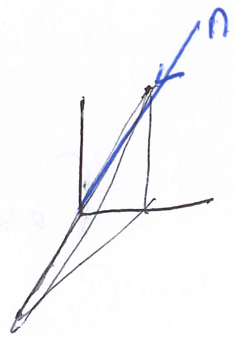
$$F = \begin{pmatrix} pA_1 \cos\alpha_1 \\ pA_1 \cos\beta_2 \\ pA_1 \cos\gamma_3 \\ pA_2 \cos\alpha_1 \\ \vdots \\ pA_3 \cos\gamma_3 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$A_1 = |\bar{n}| A_3 = \cancel{A_1} \cos\alpha\, A_3$$
$$A_2 = |\bar{n}_2| A_3 = \cos\beta\, A_3$$
$$A_4 = |\bar{n}_3| A_3 = \cos\gamma\, A_3$$

We choose $A_4 = \dfrac{x_1 y_2}{2}$

$A_1 =$





$$A_3 = \frac{A_4}{\cos\gamma} = \frac{x_1 y_2}{2\cos\gamma}$$

$$A_2 = \cos\beta \, \frac{x_1 y_2}{2\cos\gamma}$$

$$A_1 = \cos\alpha \, \frac{x_1 y_2}{2\cos\gamma}$$

$$p \, N_T \begin{pmatrix} \cos^2\alpha_1 \frac{x_1 y_2}{2\cos\gamma_1} \\ \cos\alpha_1 \cos\beta_1 \frac{x_1 y_2}{2\cos\gamma_1} \\ \cos\alpha_1 \cos\gamma_1 \frac{x_1 y_2}{2\cos\gamma_1} \\ \cos\alpha_2 \cos\beta_1 \frac{x_1 y_2}{2\cos\gamma_1} \\ \cos\beta_2 \cos\beta_1 \frac{x_1 y_2}{2\cos\gamma_1} \\ \cos\gamma_2 \cos\beta_1 \frac{x_1 y_2}{2\cos\gamma_1} \\ \cos\alpha_3 \, \cancel{\cos\beta} \frac{x_1 y_2}{2\cos\gamma_1} \\ \cos\beta_3 \frac{x_1 y_2}{2\cos\gamma_1} \\ \cos\gamma_3 \frac{x_1 y_2}{2\cos\gamma_1} \\ 0 \\ 0 \\ 0 \end{pmatrix} \frac{(x_2 y_3 z_4)^{-1}}{6} = \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix}$$

$$N_T \rho \frac{x_1 y_2}{2\cos\gamma_1} \begin{pmatrix} \cos^2\alpha_1 \\ \cos\alpha_1 \cos\beta_1 \\ \cos\alpha_1 \cos\gamma_1 \\ \cos\alpha_2 \cos\beta_1 \\ \cos\beta_2 \cos\beta_1 \\ \cos\gamma_2 \cos\beta_1 \\ \cos\alpha_3 \\ \cos\beta_3 \\ \cos\gamma_3 \\ 0 \\ 0 \\ 0 \end{pmatrix} \frac{x_2 y_3 z_4}{6} = f^e = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}$$

Forces in the most general way:

$$\rho \frac{x_1 y_2}{2\cos\gamma_1} \frac{x_2 y_3 z_4}{6} \left(\xi_1 \cos^2\alpha_1 + \xi_2 \cos\alpha_2 \cos\beta_1 + \cos\alpha_3 \xi_3 \right) = f_x$$

$$\rho \frac{x_1 y_2}{2\cos\gamma_1} \frac{x_2 y_3 z_4}{6} \left(\xi_1 \cos\alpha_1 \cos\beta_1 + \xi_2 \cos\beta_2 \cos\beta_1 + \cos\beta_3 \right) = f_y$$

$$\rho \frac{x_1 y_2}{2\cos\gamma_1} \frac{x_2 y_3 z_4}{6} \left(\xi_1 \cos\alpha_1 \cos\gamma_1 + \xi_2 \cos\gamma_2 \cos\beta_1 + \cos\gamma_3 \right) = f_z$$

If we know its coordinates we can calculate the angle, and the ~~the~~ forces