MASTER OF SCIENCE IN COMPUTATIONAL MECHANICS

COMPUTATIONAL STRUCTURAL MECHANICS AND DYNAMICS

Assignment 1:
The Direct Stiffness Method

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1. **Consider the truss problem defined in the Figure 1. All geometric and material properties: L, α, E and A, as well as the applied forces P and H are to be kept as variables. This truss has 8 degrees of freedom, with six of them removable by the fixed-displacement conditions at node 2,3 and 4. This structure is statically indeterminate as long as $\alpha \neq 0$.**

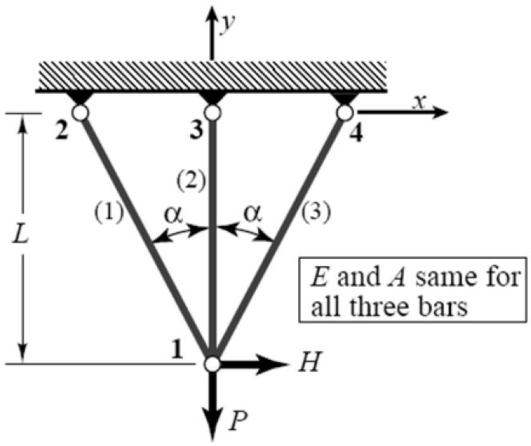

Figure 1: Truss structure. Geometry and mechanical features

(a) **Show that the master stiffness equations are:**

$$\frac{EA}{L}\begin{bmatrix} 2cs^2 & 0 & -cs^2 & c^2s & 0 & 0 & -cs^2 & -c^2s \\ & 1+2c^2 & c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \\ & & cs^2 & -c^2s & 0 & 0 & 0 & 0 \\ & & & c^3 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 \\ & & & & & & cs^2 & c^2s \\ Symm & & & & & & & c^3 \end{bmatrix} \begin{bmatrix} u_{x_1} \\ u_{y_1} \\ u_{x_2} \\ u_{y_2} \\ u_{x_3} \\ u_{y_3} \\ u_{x_4} \\ u_{y_4} \end{bmatrix} = \begin{bmatrix} H \\ -P \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

in which $c = cos(\alpha)$ and $s = sin(\alpha)$. Explain from physics why the 5th row and column contain only zeros.

To each of the truss elements, it is possible to write Hooke's Law in a matricial form as follows:

$$\bar{K}^{(e)}\bar{u}^{(e)} = \bar{f}^{(e)} \tag{2}$$

where the upper bar indicates that the variables are expressed according to a local coordinate system. Taking the local x-axis to be aligned with the truss element axis, matrix $\bar{K}^{(e)}$ can be written as:

$$\bar{K}^{(e)} = \frac{EA}{L^{(e)}}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3}$$

Note that the member index for the material properties E and A has been dropped since the problem states that their values are the same for the three bars of the truss system.

Once the individual stiffness matrices in local coordinates have been found, one must assemble them into the global system. To globalize, it is possible to prove that for a bar with an inclination angle φ with respect to the global coordinate system, the displacements satisfy the following expressions:

$$\begin{aligned} \bar{u}_{x_i} &= u_{x_i} c' + u_{y_i} s' \\ \bar{u}_{x_j} &= u_{x_j} c' + u_{y_j} s' \\ \bar{u}_{y_i} &= -u_{x_i} s' + u_{y_i} c' \\ \bar{u}_{x_i} &= -u_{x_j} s' + u_{y_j} s' \end{aligned} \tag{4}$$

where $c' = cos(\varphi)$, $s' = sin(\varphi)$ and i and j are the global numbering of the nodes. Thus, the displacements in the global system of coordinates can be expressed as follows:

$$\bar{u}^{(e)} = T^{(e)} u^{(e)} = \begin{bmatrix} \bar{u}_{x_i} \\ \bar{u}_{y_i} \\ \bar{u}_{x_j} \\ \bar{u}_{y_j} \end{bmatrix} = \begin{bmatrix} c' & s' & 0 & 0 \\ -s' & c' & 0 & 0 \\ 0 & 0 & c' & s' \\ 0 & 0 & -s' & c' \end{bmatrix} \begin{bmatrix} u_{x_i} \\ u_{y_i} \\ u_{x_j} \\ u_{y_j} \end{bmatrix} \tag{5}$$

Analogously, for the force vector the expression $f^{(e)} = (T^{(e)})^T \bar{f}^{(e)}$ holds for any truss member. As a result, the stiffness matrix of the elements in the global system of coordinates can be written as:

$$K^{(e)} = (T^{(e)})^T \bar{K}^{(e)} T^{(e)} = \frac{EA}{L^{(e)}} \begin{bmatrix} c'^2 & s'c' & -c'^2 & -s'c' \\ s'c' & s'^2 & -s'c' & -s'^2 \\ -c'^2 & -s'c' & c'^2 & s'c' \\ -s'c' & -s'^2 & s'c' & s'^2 \end{bmatrix} \tag{6}$$

For the given truss system, $\varphi^{(1)} = -(\frac{\pi}{2} - \alpha)$, $\varphi^{(2)} = \frac{\pi}{2}$ and $\varphi^{(3)} = (\frac{\pi}{2} - \alpha)$. Using trigonometric properties, the following expressions are satisfied:

$Element(1)$	$Element(2)$	$Element(3)$
$cos(\varphi^{(1)}) = sin(\alpha) \Rightarrow c' = s$	$cos(\varphi^{(2)} = 0$	$cos(\varphi^{(3)}) = sin(\alpha) \Rightarrow c' = s$
$sin(\varphi^{(1)}) = -cos(\alpha) \Rightarrow s' = -c$	$sin(\varphi^{(2)} = 1$	$sin(\varphi^{(3)}) = cos(\alpha) \Rightarrow s' = c$

Using the notation given at the beginning of the problem, the following results are obtained:

$$K^{(1)} = \frac{EAc}{L} \begin{bmatrix} s^2 & -sc & -s^2 & sc \\ -sc & c^2 & sc & -c^2 \\ -s^2 & sc & s^2 & -sc \\ sc & -c^2 & -sc & c^2 \end{bmatrix}$$

$$K^{(2)} = \frac{EA}{L} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$K^{(3)} = \frac{EAc}{L}\begin{bmatrix} s^2 & sc & -s^2 & -sc \\ sc & c^2 & -sc & -c^2 \\ -s^2 & -sc & s^2 & sc \\ -sc & -c^2 & sc & c^2 \end{bmatrix}$$

Finally, to assembly the element stiffness matrices they are augmented (considering compatibility and equilibrium conditions) and their sum yields the following expression for the global stiffness matrix:

$$\frac{EA}{L}\begin{bmatrix} 2cs^2 & 0 & -cs^2 & c^2s & 0 & 0 & -cs^2 & -c^2s \\ & 1+2c^2 & c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \\ & & cs^2 & -c^2s & 0 & 0 & 0 & 0 \\ & & & c^3 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 \\ & & & & & & cs^2 & c^2s \\ Symm & & & & & & & c^3 \end{bmatrix} \tag{7}$$

which coincides with the given system (see Equation 1). In addition, it can be noticed that both row and column 5 contain only zeros, which can be explained considering the geometry and constraints of the truss system (see Figure 1). Firstly, since node 3 is not connected to nodes 2 and 4 and its connection with node 1 is vertical, the 5th column can only contain zeros because this column represents the effects of the displacement u_x on the internal forces of each bar. Similarly, the 5th row can only contain zeros because it represents the effects of the displacements of each node on the internal force f_x at node 3.

(b) **Apply the BC's and show the 2-equation modified stiffness system**

Since the system is fixed at joints 2,3 and 4, the displacement vector satisfies the following:

$$u_{x_1} = u_{y_1} = u_{x_2} = u_{y_2} = u_{x_4} = u_{y_4} = 0$$

Thus, by eliminating the rows corresponding to displacements that are already known and the columns which would be multiplied by zero, Equation (1) can be reduced to the following system:

$$\frac{EA}{L}\begin{bmatrix} 2cs^2 & 0 \\ 0 & 1+2c^3 \end{bmatrix}\begin{bmatrix} u_{x_1} \\ u_{y_1} \end{bmatrix} = \begin{bmatrix} H \\ -P \end{bmatrix} \tag{8}$$

(c) **Solve for the displacements u_{x_1} and u_{y_1}. Check that the solution makes physical sense for the limits cases $\alpha \to 0$ and $\alpha \to \frac{\pi}{2}$. Why does u_{x_1} 'blow up' if $H \neq 0$ and $\alpha \to 0$.**

Solution of system (8) yields the following displacements:

$$u_{x_1} = \frac{HL}{2EAs^2c}$$

$$u_{y_1} = -\frac{PL}{EA(1+2c^3)}$$

The expression for u_{x_1} makes physical sense for intermediate values, although it presents issues with the limit cases. If $\alpha \to 0$ the bars would be aligned and constrained at one point (in this case, node 3). Besides, if in the addition the condition $H = 0$ is imposed, there would be no resistance for the motion and equilibrium could not be possible. As a result the bars would rotate and the solution would 'blow up'.Similarly, if $\alpha \to \frac{\pi}{2}$, bars 1 and 2 would have infinite length and once again the solutions makes no physical sense.

On the other hand, the expression for u_{y_1} stays mathematically possible for all values of α, although as mentioned before, for $\alpha \to 0$ the bars would be aligned together and, in the case of $\alpha \to \frac{\pi}{2}$, the displacement on the y-direction will reach its highest value, but it makes no physical sense since bars 1 and 2 would need to be of infinite length.

(d) **Recover the axial forces in the members. Partial answers:** $F^{(3)} = -H/(2s)+Pc^2/(1+2c^3)$**. Why do** $F^{(1)}$ **and** $F^{(3)}$ **'blow up' if** $H \neq 0$ **and** $\alpha \to 0$**?**

The axial force F in each one of truss elements can be found using the following expression:

$$F^{(e)} = \frac{E^{(e)}A^{(e)}}{L^{(e)}}(\bar{u}_{xj}^{(e)} - \bar{u}_{xi}^{(e)}) \tag{9}$$

To return to a local systems of coordinates expression (5) can be used. Thus,

$$\bar{u}^{(1)} = T^{(1)}u^{(1)} = \begin{bmatrix} \bar{u}_{x_1}^{(1)} \\ \bar{u}_{y_1}^{(1)} \\ \bar{u}_{x_2}^{(1)} \\ \bar{u}_{y_2}^{(1)} \end{bmatrix} = \begin{bmatrix} s & -c & 0 & 0 \\ c & s & 0 & 0 \\ 0 & 0 & s & -c \\ 0 & 0 & c & s \end{bmatrix} \begin{bmatrix} u_{x_2} = 0 \\ u_{y_2} = 0 \\ u_{x_1} \\ u_{y_1} \end{bmatrix}$$

$$\bar{u}^{(2)} = T^{(2)}u^{(2)} = \begin{bmatrix} \bar{u}_{x_1}^{(2)} \\ \bar{u}_{y_1}^{(2)} \\ \bar{u}_{x_2}^{(2)} \\ \bar{u}_{y_2}^{(2)} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} u_{x_1} \\ u_{y_1} \\ u_{x_3} = 0 \\ u_{y_3} = 0 \end{bmatrix}$$

$$\bar{u}^{(3)} = T^{(3)}u^{(3)} = \begin{bmatrix} \bar{u}_{x_1}^{(3)} \\ \bar{u}_{y_1}^{(3)} \\ \bar{u}_{x_2}^{(3)} \\ \bar{u}_{y_2}^{(3)} \end{bmatrix} = \begin{bmatrix} s & c & 0 & 0 \\ -c & s & 0 & 0 \\ 0 & 0 & s & c \\ 0 & 0 & -c & s \end{bmatrix} \begin{bmatrix} u_{x_1} \\ u_{y_1} \\ u_{x_4} = 0 \\ u_{y_4} = 0 \end{bmatrix}$$

As a result the elongation for each bar will be:

$$d^{(1)} = (\bar{u}_{x2}^{(1)} - \bar{u}_{x1}^{(1)}) = (su_{x1} - cu_{y1}) - 0 = \frac{HL}{2EAcs} + \frac{PLc}{EA(1+2c^3)}$$

$$d^{(2)} = (\bar{u}_{x2}^{(2)} - \bar{u}_{x1}^{(2)}) = 0 - u_{y1} = \frac{PL}{EA(1+2c^3)}$$

$$d^{(3)} = (\bar{u}^{(2)}_{x2} - \bar{u}^{(2)}_{x1}) = 0 - (su_{x1} + cu_{y1}) = -\frac{HL}{2EAcs} + \frac{PLc}{EA(1+2c^3)}$$

Thus applying equation (9), the axial forces yield:

$$F^{(1)} = \tfrac{H}{2s} + \tfrac{Pc^2}{1+2c^3} \quad F^{(2)} = \tfrac{P}{1+2c^3} \quad F^{(3)} = -\tfrac{H}{2s} + \tfrac{Pc^2}{1+2c^3}$$

If the equations for $F^{(1)}$ and $F^{(3)}$ are analyzed, it can be seen that the solution 'blows up' in the case of $\alpha \to 0$. Physically, for this limit case the equilibrium condition is not satisfied and also no axial force can compensate the horizontal force ($H \neq 0$). Consequently, the equilibrium solution will tend to infinite while the bars will start to rotate about node 3.

2. **Dr. Who proposes 'improving' the result for the example truss of the 1st lesson by putting one extra node, 4 at the midpoint of member (3) 1-3, so that it is subdivided in two different members: (3) 1-4 and (4) 3-4. His 'reasoning' is that more is better. Try Dr. Who's suggestion by hand computations and verify that the solution 'blows up' because the modified master stiffness is singular. Explain physically.**

For this particular problem, the following system:

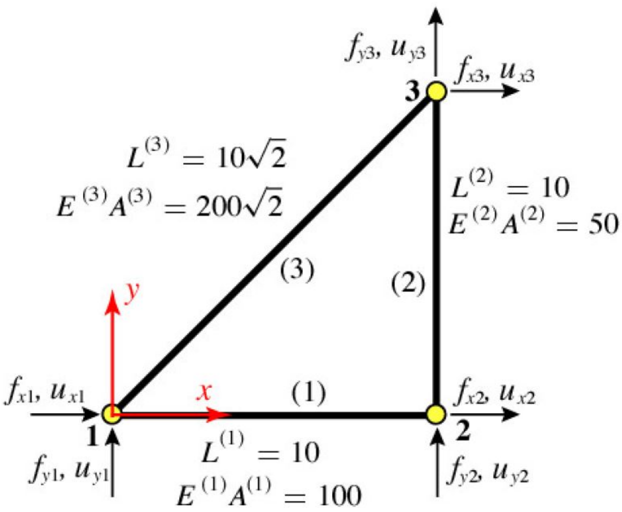

Figure 2: Original truss system

will be modified to have an extra node (node 4) at the midpoint of element (3). Dividing the former element (3) into the new elements (3) and (4) will result in the the following geometric and material properties:

$$L^{(3)} = L^{(4)} = 5\sqrt{2}$$
$$E^{(3)}A^{(3)} = E^{(4)}A^{(4)} = 200\sqrt{2}$$

Following the same procedure as the one used in the previous exercise, the element stiffness matrices in the global system of coordinates can be obtained:

$$\bar{K}^{(1)} = \begin{bmatrix} 10 & 0 & -10 & 0 \\ 0 & 0 & 0 & 0 \\ -10 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\bar{K}^{(2)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & -5 & 0 & 5 \end{bmatrix}$$

$$\bar{K}^{(3)} = \bar{K}^{(4)} = \begin{bmatrix} 20 & 20 & -20 & -20 \\ 20 & 20 & -20 & -20 \\ -20 & -20 & 20 & 20 \\ -20 & -20 & 20 & 20 \end{bmatrix}$$

Assembling the previously obtained element matrices using expanded versions of them, the global stiffness matrix has the following form:

$$\begin{bmatrix} 30 & 20 & -10 & 0 & 0 & 0 & -20 & -20 \\ & 20 & 0 & 0 & 0 & 0 & -20 & -20 \\ & & 10 & 0 & 0 & 0 & 0 & 0 \\ & & & 5 & 0 & -5 & 0 & 0 \\ & & & & 20 & 20 & -20 & -20 \\ & & & & & 25 & -20 & -20 \\ & & & & & & 40 & 40 \\ Symm & & & & & & & 40 \end{bmatrix} \tag{10}$$

As it can be seen, the last two rows of the matrix are equal (or a linear combination of themselves), therefore the system cannot be solved. Physically, this means that two new degrees of freedom are introduced with no additional boundary conditions. Furthermore, the added two degrees of freedom contradict the initial assumption of element (3) being an truss member, i.e. a straight line. The system could be solved only if the displacements of node 4 are constrained in such a way that node 1,3,4 will remain collinear, i.e. adding a new compatibility condition.