Assignment 1 Computational Structural Mechanics and Dynamics, Assignment 1

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On the "Direct Stiffness Method":

Consider the truss problem defined in the figure. All geometric and material properties: L, α, E, and A, as well as the applied forces are to be kept as variables. This truss has 8 degrees of freedom, with 6 of them removable by the fixed-displacement conditions at nodes 2, 3, and 4. This structure is statically indeterminate as long as $\alpha \neq 0$.

(a) Show that the master stiffness equations are:

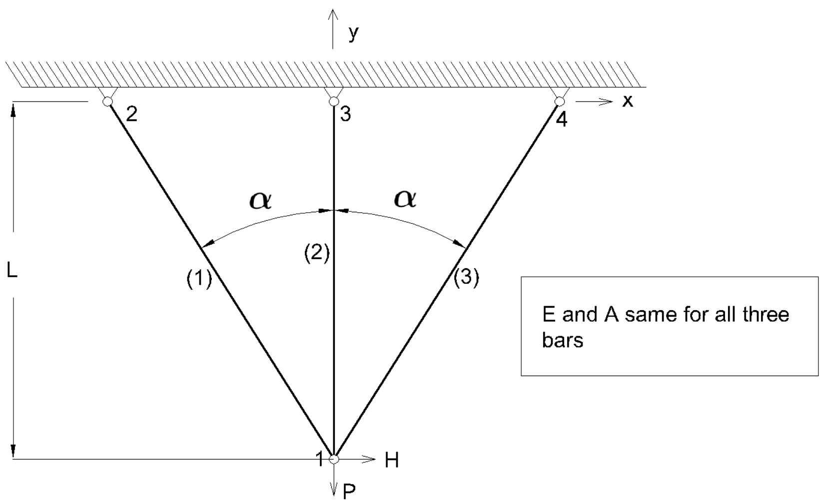
$$\frac{EA}{L}\begin{bmatrix} 2cs^2 & 0 & -cs^2 & c^2s & 0 & 0 & -cs^2 & -c^2s \\ & 1+2c^3 & c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \\ & & cs^2 & -c^2s & 0 & 0 & 0 & 0 \\ & & & c^3 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 \\ & & & & & & cs^2 & c^2s \\ symm & & & & & & & c^3 \end{bmatrix}\begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix} = \begin{bmatrix} H \\ -P \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

in which c= cos α and s= sin α. Explain from physics why the 5th row and column contain only zeros.

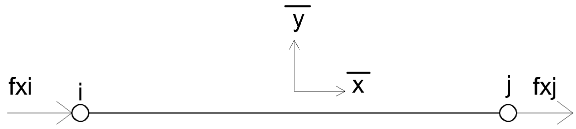
(b) Apply the BCs and show the 2-equation modified stiffness system.

(c) Solve for the displacements u_{x1} and u_{y1}. Check that the solution makes physical sense for the limit cases $\alpha \to 0$ and $\alpha \to \pi/2$. Why does the solution "blow up" if $H \neq 0$ and $\alpha \to 0$

(d) Recover the axial forces in the three memebers. Partial answer $F^{(3)} = -H/2s + Pc^2/(1+2c^3)$. Why do $F^{(1)}$ and $F^{(3)}$ "blow up" if $H \neq 0$ and $\alpha \to 0$?



To answer point (a), one needs to first start with the reference element and matrix



$$\underbrace{\frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Matrix } \overline{K}} \begin{bmatrix} u_{xi} \\ u_{yi} \\ u_{xj} \\ u_{yj} \end{bmatrix} = \begin{bmatrix} f_{xi} \\ f_{yi} \\ f_{xj} \\ f_{yj} \end{bmatrix}$$

Using the appropriate transformations (Rotation Matrix T), one can use this generic matrix $\overline{K}$, to suit all the bars. It is important to notice that the correct rotation angle is used, that is $(\pi/2) - \alpha$, and $(\pi/2) + \alpha$ respectively. The following rotation matrices are used for each bar, according to their label 1, 2, and 3.

$$T_1 = \begin{bmatrix} s & -c & 0 & 0 \\ c & s & 0 & 0 \\ 0 & 0 & s & c \\ 0 & 0 & c & s \end{bmatrix}$$

$$T_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$T_3 = \begin{bmatrix} s & c & 0 & 0 \\ -c & s & 0 & 0 \\ 0 & 0 & s & c \\ 0 & 0 & -c & s \end{bmatrix}$$

From these rotation matrices, the only one with fixed values is T_2, since it is the only known angle at $\pi/2$ rad.

After performing the rotation of the matrices, using $K = (T^T)\overline{K}T$, the resulting matrices are:

$$K_1 = \frac{EA}{L}\begin{bmatrix} cs^2 & -c^2s & -cs^2 & c^2s \\ -c^2s & c^3 & c^2s & -c^3 \\ -cs^2 & c^2s & cs^2 & -c^s2 \\ c^2s & -c3 & -c^2s & c^3 \end{bmatrix}$$

$$K_2 = \frac{EA}{L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$K_3 = \frac{EA}{L}\begin{bmatrix} cs^2 & c^2s & -cs^2 & -c^2s \\ c^2s & c^3 & -c^2s & -c^3 \\ -cs^2 & -c^2s & cs^2 & c^s2 \\ -c^2s & -c3 & c^2s & c^3 \end{bmatrix}$$

Which results, after assembling the global matrix in:

$$\frac{EA}{L}\begin{bmatrix} 2cs^2 & 0 & -cs^2 & c^2s & 0 & 0 & -cs^2 & -c^2s \\ & 1+2c^3 & c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \\ & & cs^2 & -c^2s & 0 & 0 & 0 & 0 \\ & & & c^3 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 \\ & & & & & & cs^2 & c^2s \\ symm & & & & & & & c^3 \end{bmatrix}\begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix} = \begin{bmatrix} H \\ -P \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \\ f_{x4} \\ f_{y4} \end{bmatrix}$$

The matrix is the same that one is supposed to obtain, but notice that the forcing term is being left undefined as it should be. In the problem statement is was a vector with most of its entries as zeroes. Moreover, the fifth row and column contain only zeros because they correspond to the reaction force in the x-direction on node 3. Notice that because of the vertical position bar (2) has, this node will not produce any reactions. This is true since the original configuration is being studied. After some deformation, when bar (2) tilted a bit, some reaction can appear in the x-direction, but thats not being considered here.

(b)After applying the BCs, the following system is obtained:

$$\frac{EA}{L}\begin{bmatrix} 2cs^2 & 0 \\ 0 & 1+2c^3 \end{bmatrix}\begin{bmatrix} u_{x1} \\ u_{y1} \end{bmatrix} = \begin{bmatrix} H \\ -P \end{bmatrix}$$

(c) The solution of the reduced system is:

$$U_{x1} = \frac{L}{EA}\frac{H}{cs^2} \qquad U_{y1} = \frac{L}{EA}\frac{-P}{2c^3+1}$$

Knowing these displacements, it is possible to retrieve the reaction forces as:

$$\frac{EA}{L}\begin{bmatrix} 2cs^2 & 0 & -cs^2 & c^2s & 0 & 0 & -cs^2 & -c^2s \\ & 1+2c^3 & c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \\ & & cs^2 & -c^2s & 0 & 0 & 0 & 0 \\ & & & c^3 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 \\ & & & & & & cs^2 & c^2s \\ symm & & & & & & & c^3 \end{bmatrix}\begin{bmatrix} \frac{L}{EA}\frac{H}{cs^2} \\ \frac{L}{EA}\frac{-P}{2c^3+1} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} H \\ -P \\ -\frac{H}{2} - \frac{Pc^2s}{2c^3+1} \\ \frac{Pc^3}{2c^3+1} + \frac{Hc}{2s} \\ 0 \\ \frac{P}{2c^3+1} \\ \frac{Pc^2s}{2c^3+1} - \frac{H}{2} \\ \frac{Pc^3}{2c^3+1} - \frac{Hc}{2s} \end{bmatrix}$$

Now, since the term involving the displacement in x direction of the node 1 (U_x1) contains both sin and cosine in the denominator, the two limit cases 0 and $\pi/2$ will make the displacement to increase too much. From physics, it can be said that if the angle is 0, there is only one bar, which cannot withstand any moment. If the angle is $\pi/2$ the bars are infinitely apart.

(d) In order to obtain the axial forces on the members, one need to recover the displacements on the nodes of each bar, and rotate them back using ($\overline{U} = TU$). Then, one finds the deformation induced by ($d = \overline{U}_{x2} - \overline{U}_{x1}$), and after multiplying it by EA/L, one obtains the axial force.

The axial forces are then:

$$F^{(3)} = -\frac{H}{2s} + \frac{Pc^2}{2c+1}$$

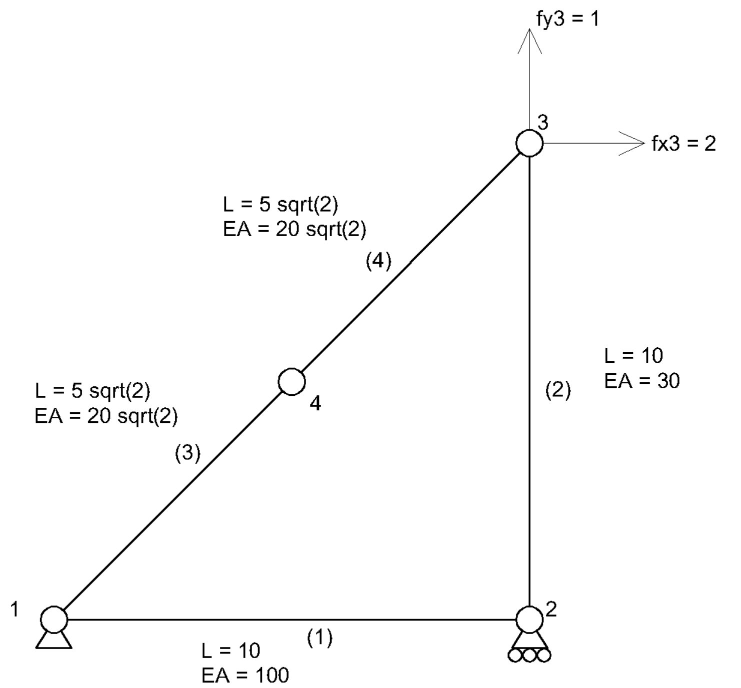
$$F^{(2)} = \frac{Pc}{2c^3+1}$$

$$F^{(1)} = -\frac{H}{2s} - \frac{Pc^2}{2c+1}$$

Similarly to the explanation of the last point, when $\alpha \to 0$, the structures converges into a single bar, which is not capable of withstanding moments, therefore, no H would be allowed.

Assignment 2

Dr. Who proposes "improving" the result from the example truss of the 1^{st} lesson by putting one extra node, 4, at the midpoint of the member (3) 1-3, so that it is subdivided in two different members: (3) 1-4 and (4) 3-4. His reasoning is that more is better. Try Dr. Whos suggestion by hand computations and verify that the solution "blows up" because the modified stiffness matrix is singular. Explain physically.



Following the steps already mentioned in the first part of this report, the matrices corresponding to each of the members are:

$$K_1 = 10 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K_2 = 5 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$K_3 = K_4 = 40 \begin{bmatrix} .5 & .5 & -.5 & -.5 \\ .5 & .5 & -.5 & -.5 \\ -.5 & -.5 & .5 & .5 \\ -.5 & -.5 & .5 & .5 \end{bmatrix}$$

Which are afterwards assembled into the master stiffness system:

$$\begin{bmatrix} 30 & 20 & -10 & 0 & 0 & 0 & -20 & -20 \\ & 20 & 0 & 0 & 0 & 0 & -20 & -20 \\ & & 10 & 0 & 0 & 0 & 0 & 0 \\ & & & 5 & 0 & -5 & 0 & 0 \\ & & & & 20 & 20 & -20 & -20 \\ & & & & & 25 & -20 & -20 \\ & & & & & & 40 & 40 \\ symm & & & & & & & 40 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix} = \begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ 2 \\ 1 \\ f_{x4} \\ f_{y4} \end{bmatrix}$$

And after imposing the BCs:

$$\begin{bmatrix} 10 & 0 & 0 & 0 & 0 \\ & 20 & 20 & -20 & -20 \\ & & 25 & -20 & -20 \\ & & & 40 & 40 \\ symm & & & & 40 \end{bmatrix} \begin{bmatrix} u_{x2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix} = \begin{bmatrix} f_{x2} \\ 2 \\ 1 \\ f_{x4} \\ f_{y4} \end{bmatrix}$$

When inserting a new node, one also inserted new unfixed DOF, (Ux4 and Uy4); these make it impossible to solve the system since now the resulting matrix is singular, unlike in the class problem.
From physics, it can be seen that the frame member does not deform freely in the middle of it, so the modeling of the physics is not correct when adding this extra node in the middle of the member.
Moreover, when modeling in 2D, one needs to make sure to suppress the rotations and translations, and in the example with the extra node, the new DOFs are free, and the structure can be considered as a mechanism.