Computational Structural Mechanics and Dynamics

Masters in Numerical Methods

Assignment 3

Plane stress Problem and Linear Triangle

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1 Assignment 3.1

Question 1

For such a problem we choose a triangular element with a plane stress case. The coordinates are given as follows, $x_1 = 0$,$y_1 = 0$,$x_2 = 1$,$y_2 = 0$ and $x_3 = 0$, $y_3 = 1$.
The shape functions are given by

$$N_i = \frac{1}{2A^e}(a_i + b_i x + c_j y) \tag{1}$$

for $i = 1, 2, 3$. Where A^e is the element area. $a_i = x_j y_k - x_k y_j$, $b_i = y_j - y_k$, $c_i = x_k - x_j$, here $i, j, k = 1, 2, 3$.
The co-ordinates are given by

$$X = \begin{bmatrix} 0 & 3 & 3 \\ 0 & 1 & 2 \end{bmatrix} \tag{2}$$

The stiffness matrix is given by the equation

$$K_{ij}^e = \int\int_{A^e} B_i^T D B_j t dA \tag{3}$$

The stresses are defined as

$$\sigma = \frac{E}{1-\nu}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}\epsilon \tag{4}$$

The stress matrix given for this particular problem is as follows, $D = \begin{bmatrix} 100 & 25 & 0 \\ 25 & 100 & 0 \\ 0 & 0 & 50 \end{bmatrix}$ in this case and the thickness is 1.
The matrix $B_i = \begin{bmatrix} b_i & 0 \\ 0 & c_i \\ c_i & b_i \end{bmatrix}$
The stiffness matrix is symmetric and it is given by

$$K = \begin{bmatrix} 18.75 & 9.375 & -12.5 & -6.25 & -6.25 & -3.125 \\ 9.375 & 18.75 & 6.25 & 12.5 & -15.625 & -31.25 \\ -12.5 & 6.25 & 75 & -37.5 & -62.5 & 31.25 \\ -6.25 & 12.5 & -37.5 & 75 & 43.75 & -87.5 \\ -6.25 & -15.625 & -62.5 & 43.75 & 68.75 & -28.125 \\ -3.125 & -31.25 & 31.25 & -87.5 & -28.125 & 118.75 \end{bmatrix} \tag{5}$$

Question 2

It can be seen that the sum of rows and columns of 1,3, and 5 as well as rows and columns 2,4 and 6 vanish because individually these represent the equilibrium in the X and Y directions of this 2D element. Since the triangular element is in equilibrium we can see that the rows and columns of the stiffness matrix disappear.

2 Assignment 3.2

Question a

Calculate the stiffness matrices for both the elements.

For the first case we use the same equation (3) as in the first case, only in this particular case the Poisson ratio $\nu = 0$. The stress matrix is given by

$$\sigma = E \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} \epsilon \tag{6}$$

The co-ordinates are given by The co-ordinates are given by

$$X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{7}$$

The stiffness matrix is then given by

$$K_{triangle} = E \begin{bmatrix} 0.75 & 0.25 & -0.5 & -0.25 & -0.25 & 0 \\ 0.25 & 0.75 & 0 & -0.25 & -0.25 & -0.5 \\ -0.5 & 0 & 0.5 & 0 & 0 & 0 \\ -0.25 & -0.25 & 0 & 0.25 & 0.25 & 0 \\ -0.25 & -0.25 & 0 & 0.25 & 0.25 & 0 \\ 0 & -0.5 & 0 & 0 & 0 & 0.5 \end{bmatrix} \tag{8}$$

here the length and the thickness are both taken to be one.

For a set of three bar elements the stiffness matrix can we written as

$$K = \frac{EA^e}{L^e} \begin{bmatrix} c^2 & sc & -c^2 & -cs \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & cs \\ -sc & -s^2 & sc & s^2 \end{bmatrix} \tag{9}$$

Here, s and c are the sine and cosine of the angle the bar makes with respect to the horizontal. For the given problem, $\alpha = \frac{\pi}{2}, 0$ and $-\frac{\pi}{4}$ for respectively for the three elements.

Thus, the element stiffness matrix are given by

$$K_1 = \frac{EA_1}{a}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \tag{10}$$

$$K_2 = \frac{EA_2}{a}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{11}$$

$$K_3 = \frac{EA_3}{\sqrt{a}}\begin{bmatrix} 0.5 & -0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix} \tag{12}$$

Considering $A_1 = A_2 = A$ and $A_3 = A'$ and the length of a segment a $= 1$.

$$K_{bar} = E\begin{bmatrix} A & 0 & -A & 0 & 0 & 0 \\ 0 & A & 0 & 0 & 0 & -A \\ -A & 0 & A + \frac{A'}{2\sqrt{2}} & -\frac{A'}{2\sqrt{2}} & -\frac{A'}{2\sqrt{2}} & \frac{A'}{2\sqrt{2}} \\ 0 & 0 & -\frac{A'}{2\sqrt{2}} & \frac{A'}{2\sqrt{2}} & \frac{A'}{2\sqrt{2}} & -\frac{A'}{2\sqrt{2}} \\ 0 & 0 & -\frac{A'}{2\sqrt{2}} & \frac{A'}{2\sqrt{2}} & \frac{A'}{2\sqrt{2}} & -\frac{A'}{2\sqrt{2}} \\ 0 & -A & \frac{A'}{2\sqrt{2}} & -\frac{A'}{2\sqrt{2}} & -\frac{A'}{2\sqrt{2}} & A + \frac{A'}{2\sqrt{2}} \end{bmatrix} \tag{13}$$

Question b

The two matrices are not similar to each other as the matrix obtained from the triangular element has many non-zero entries. The two matrices represent different structural problems, one is a triangle as a whole and the other is truss structure of three bars. To make the elements in the diagonal as similar as possible we can choose the values of $A = 0.5$ and $A' = \frac{1}{sqrt2}$. This will cause a few diagonal elements to be similar.

Question c

The reason why the bar structure matrix has fewer non-zero elements is because in that case the loads are exclusively applied to the nodes. In case of the triangle element structure the loads can be applied to nodes as well as the boundary. Thus this matrix has more non-zero entries.

Question d

If the Poisson ratio is non zero the stiffness matrix for the bar element will change as follows.

$$K_{bar} = \frac{E}{1-\nu^2}\begin{bmatrix} 0.75-\frac{\nu}{2} & 0.25+\frac{\nu}{4} & -0.5 & -0.25-\frac{\nu}{4} & \frac{\nu}{4}-0.25 & -\frac{\nu}{2} \\ 0.25+\frac{\nu}{4} & 0.75-\frac{\nu}{4} & -\frac{\nu}{2} & \frac{\nu}{4}-0.25 & \frac{\nu}{4}-0.25 & -0.5 \\ -0.5 & -\frac{\nu}{2} & 0.5 & 0 & 0 & \frac{\nu}{2} \\ \frac{\nu}{4}-0.25 & \frac{\nu}{4}-0.25 & 0 & -\frac{\nu}{4}+0.25 & 0.25-\frac{\nu}{4} & 0 \\ \frac{\nu}{4}-0.25 & \frac{\nu}{4}-0.25 & 0 & -\frac{\nu}{4}+0.25 & 0.25-\frac{\nu}{4} & 0 \\ \frac{\nu}{2} & -0.5 & \frac{\nu}{2} & 0 & 0 & 0.5 \end{bmatrix} \tag{14}$$

The Poisson ratio relates the strains in different dimensions to the stresses when $\nu = 0$, in a plane stress case we see that the D matrix is diagonal. Which means that the stress in one dimension causes strain in that dimension. When the Poisson ratio is non-zero a stress in one dimension can cause strains in various other dimensions. Which is why we see that the Stiffness matrix obtained in this case has fewer non-zero elements.