Federico Valencia Otálvaro

FEF Midterm

Exercise 2

$$\begin{cases} -\nu \dfrac{\partial^2 u}{\partial x^2} + \beta \dfrac{\partial u}{\partial x} = 0 & -1 \le x \le 1 \\ u = 0 & x = -1 \\ u = -1 & x = 1 \end{cases}$$

$\nu = 0{,}03$ diff. coef

$\beta = 1{,}8$ convec. vel.

a.) Weak form

$$\int_\Omega w(\beta u_x)\, d\Omega - \int_\Omega w(\nu u_{xx})\, d\Omega = 0$$

$$\int_\Omega w(\beta u_x)\, d\Omega - \left[\int_{\partial\Omega} w(\nu u_x)\cdot n\, d\Gamma - \int_\Omega w_x(\nu u_x)\, d\Omega \right] = 0$$

$$\beta \int_\Omega w\, u_x\, d\Omega + \nu \int_\Omega w_x\, u_x\, d\Omega = \nu \int_{\partial\Omega} w(u_x \cdot n)\, d\Gamma$$

$u = u_i N_i(x) \longrightarrow u_x = u_i N_x(x)$

Galerkin $\longrightarrow w_i = N_i$ $\begin{cases} N_1 = \dfrac{x_2 - x}{h} \\ N_2 = \dfrac{x - x_1}{h} \end{cases}$ linear 1D elem.

$$\Big[\underbrace{\beta \int_\Omega N_i N_{x_j}\, d\Omega}_{\substack{c \\ \text{(non-symmetric)}}} + \underbrace{\nu \int_\Omega N_{x_i} N_{x_j}\, d\Omega}_{\substack{a \\ \text{(symmetric)}}} \Big] u_j = \underbrace{\int_{\Gamma_N} N_i\, u_N\, d\Gamma}_{\text{fluxes}}$$

$\begin{cases} N_{1x} = -\dfrac{1}{h} \\ N_{2x} = \dfrac{1}{h} \end{cases}$

For the n^{th} element $\longrightarrow a_{ij}^{(e)} = 0{,}03 \displaystyle\int_{x_1^{(e)}}^{x_2^{(e)}} \frac{1}{h^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} dx$

$(a + c)u = f$

$$c_{ij}^{(e)} = 1{,}8 \int_{x_1^{(e)}}^{x_2^{(e)}} \frac{1}{h^2} \begin{bmatrix} x - x_2 & x_2 - x \\ x_1 - x & x - x_1 \end{bmatrix} dx$$

integrate using Gauss quadrature

$$f_i^{(e)} = \int_{\Gamma_N} \begin{bmatrix} u_{N_1} \\ u_{N_2} \end{bmatrix} d\Gamma$$

b.) $N = 20 \longrightarrow h = \frac{1-(-1)}{20} = 0{,}1$ Peclet Number $\longrightarrow Pe = \frac{|a|h}{2\nu} = \frac{1.8(0{,}1)}{2(0{,}03)}$

$\underline{Pe = 3 > 1}$

Since the scheme has $Pe > 1$, the standard Galerkin FEM formulation will present numerical instabilities due to lack of sufficient diffusion in the problem

Consistent stabilization $\longrightarrow$ Add stabilization term $\sum_e \int_{\Omega_e} P(w)\tau R(u) d\Omega$

For SGS $\longrightarrow P(w) = -\mathcal{L}^*(w) = a\cdot\nabla w + \nabla\cdot(\nu\nabla w) - \cancel{\sigma w}$

$$\tau = \frac{h}{2a}\left(\coth(Pe) - \frac{1}{Pe}\right)$$

$$R(u) = a\cdot\nabla u - \nabla\cdot(\nu\nabla u) + \cancel{\sigma u - s}$$

$$\int_{\Omega_e} P(w)\tau R(u) d\Omega = \tau\int_{\Omega_e}[\beta w_x + \underbrace{\nabla\cdot(\nu w_x)}_{\text{linear elements}}][\beta u_x - \cancel{\nabla\cdot(\nu u_x)}]d\Omega = \tau\beta^2\int_{\Omega_e} w_x u_x d\Omega$$

$$= \tau\beta^2 \underbrace{\int N_{x_i} N_{x_j} d\Omega}_{a} u_j$$

The weak form becomes $\longrightarrow$

$$\boxed{\left[\beta\int_\Omega N_i N_{x_j} d\Omega + \nu\int_\Omega N_{x_i} N_{x_j} d\Omega\right] u_j + \sum_e \tau\beta^2\int_{\Omega_e} N_{x_i} N_{x_j} d\Omega\, u_j = \int_{\Gamma_N} N_i u_N d\Gamma}$$

Due to the added diffusion, this method should $\underline{\underline{\text{not}}}$ present numerical instabilities

c.) Consistent stabilization consists of adding diffusion along the streamlines in a consistent manner in order to ensure that the solution of the differential equation is also a solution of the weak form.

The stabilization parameter τ is a function of a, ν, and h. As the Peclet number decreases, the propagation of information on the FEM scheme improves and less added diffusion is necessary, an therefore, as $Pe \longrightarrow 0$, τ should vanish.

Federico Valverde Galván

Exercise 1

$$u_t + u\,u_x = 0 \qquad 0 \le x \le 1 \qquad t > 0$$

$$u = u_0(x) \qquad 0 \le x \le 1 \qquad t = 0$$

$$u = 1 \qquad x = 0 \qquad t > 0$$

$$u_0(x) = \begin{cases} 1 & 0 \le x \le p \\ 1 - \dfrac{x-p}{q-p} & p \le x \le q \\ 0 & q \le x \le 1 \end{cases}$$

$$p = 0{,}64 \qquad q = 0{,}84$$

a) 2nd Order Taylor Galerkin with constant representation of non linear flux

$$\frac{\Delta u}{\Delta t} = u_t^n + \frac{\Delta t}{2} u_{tt}^n = -f_x(u^n) + \frac{\Delta t}{2}\left[a(u^n)\, f_x(u^n)\right]_x$$

$$a(u) = \frac{\partial f}{\partial u} = u$$

$$\frac{\Delta u}{\Delta t} = -f_x(u^n) + \frac{\Delta t}{2}\left(u^n f_x(u^n)\right)_x = -f_x^n + \frac{\Delta t}{2}\left(u^{n^2} u_x^n\right)_x$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} = u\,u_x$$

$$\int_\Omega w \frac{\Delta u}{\Delta t} d\Omega = -\left[\int_{\partial\Omega} w f^n d\Gamma - \int_\Omega w_x f^n d\Omega\right] + \frac{\Delta t}{2}\left[\int_{\partial\Omega} w\, u^{n^2} u_x^n d\Gamma - \int_\Omega w_x u^{n^2} u_x^n d\Omega\right]$$

$$\int_\Omega w \frac{\Delta u}{\Delta t} d\Omega = \int_\Omega w_x f^n d\Omega - \frac{\Delta t}{2}\int_\Omega w_x u^{n^2} u_x^n d\Omega - \int_{\partial\Omega} w f^n d\Gamma + \frac{\Delta t}{2}\int_{\partial\Omega} w\, u^{n^2} u_x^n d\Gamma$$

$$\boxed{\int_0^1 w \frac{\Delta u}{\Delta t} dx = \int_0^1 w_x \left(f^n - \frac{\Delta t}{2} u^{n^2} u_x^n\right) dx - \left[w\left(f^n - \frac{\Delta t}{2} u^{n^2} u_x^n\right)\right]_0^1}$$

constant flux representation $\longrightarrow \underbrace{f(u^n) \approx f(\bar{u})}_{\text{element-wise approximation}}$ $\quad \bar{u}$: mean value of u in the element

c) Alternative choices to represent non-linear fluxes:

- Classical $\longrightarrow f(u) \approx f(N_i(x) u_i)$ (FEM approximation)
  - $\hookrightarrow$ Analytical expression integrated with Gauss quadrature
- Group Representation $\longrightarrow f(u) \approx N_i(x) f(u_i)$