Final exam Finite Elements in Fluids

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Exercise 2:

$$\begin{cases} \dfrac{\delta u}{\delta t} - \nu \nabla^2 u + (u \cdot \nabla) u + \sigma u + \nabla p = f \\ \nabla \cdot u = 0 \\ u = 0 \\ u = u_0 \end{cases}$$

Crank-Nicolson $\Rightarrow$ $\theta = \dfrac{1}{2}$, implicit

a) Time discrete problem

Theta methods: $\dfrac{v^{n+1} - v^n}{\Delta t} = \theta v_t^{n+1} + (1-\theta) v_t^n =$

$= \theta (v_t^{n+1} - v_t^n) + v_t^n$

* Momentum conservation:

$$\begin{cases} \dfrac{v^{n+1} - v^n}{\Delta t} + C(v) + K(v^{n+1/2}) + R(v^{n+1/2}) + \\ + \nabla p^{n+1} = f^{n+1/2} \\ \nabla \cdot v^{n+1} = 0 \end{cases}$$

Being,

$$C(v) = \left[(v \cdot \nabla) v\right]^{n+1/2} \Rightarrow \text{treated as implicit}$$

$$K(v^{n+1/2}) = -\nu \nabla^2 u^{n+1/2}$$

$$R(v^{n+1/2}) = \sigma u$$

b) Weak form:

$$\begin{cases} \int_\Omega w \cdot v_t \, d\Omega + \int_\Omega (\nabla w) : (\nu \nabla v) \, d\Omega + \int_\Omega w (u \cdot \nabla) u + \\ + \int_\Omega w \, \sigma \, v \, d\Omega - \int (\nabla \cdot w) \, p \, d\Omega = \int_\Omega w f + \underbrace{\int_{\Gamma_N} w \cdot t \, d\Gamma}_{\text{No Neumann condition}} \\ \int q \, \nabla \cdot v \, d\Omega = 0 \qquad \text{with } v(0) = v_0 \end{cases}$$

* We define:

$$v^h = u^h(t) + v^h_D(t)$$

$\hookrightarrow$ Dirichlet part

$$+\int w\,\sigma\,v\,d\Omega$$

$$\begin{cases} (w^h, u_t^h) + a(w^h, u^h) + c(v^h; w^h, u^h) + b(w^h, p^h) = \\ \quad = (w^h, f^h) + (w^h, t^h)_{\Gamma_N} - a(w^h, v_D^h) - c(v^h; w^h, v_D^h) \\ b(u^h, q^h) = -b(v_D^h, q^h) \end{cases}$$

with $v^h(0) = v_0^h$

* Spaces:

- Trial solution:

$$S = \{ v \in H^1(\Omega) \mid v = v_D \text{ on } \Gamma_D \}$$

- Weighting functions of velocity w

$$V = \{ w \in H^1(\Omega) \mid w = 0 \text{ on } \Gamma_D \}$$

- Space for the pressure

$$Q = L_2(\Omega)$$

\# $a(w, u) = \int_\Omega \nabla w : \nu \nabla u \, d\Omega$

$b(w, p) = -\int p \, \nabla \cdot w \, d\Omega$

c) Discrete Weak form

Introducing:

From $v_i^h = u_i^h + v_{D_i}^h$

$$u_i^h(x) = \sum_{A \in \eta / \eta_D} N_A(x)\, u_{iA}$$

$$v_{D_i}^h(x) = \sum N_A(x)\, v_{Di}(x_A)$$

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$$w_i^h \in V_i^h = \underset{A \in \eta \setminus \eta_{D_i}}{\text{span}} \{N_A\}$$

+ For pressure

$$p^h(x) = \sum_{\hat{A} \in \hat{\eta}} \hat{N}_{\hat{A}}(x)\, p_{\hat{A}}$$

$$q^h \in Q^h = \underset{\hat{A} \in \hat{\eta}}{\text{span}} \{\hat{N}_{\hat{A}}\}$$

+ Discretizing the weak form:

$$\begin{cases} M\dot{u}(t) + \left[K + C(v(t)) + R\right] u(t) - G p(t) = f(t, v(t)) \\ G^T u(t) = h(t) \\ u(0) = v_0 - v_D(0) \end{cases}$$

+ As there is no Neumann condition and the Dirichlet part is equal to 0 $\Rightarrow$ $[h = 0]$

And it is not considered the terms:

$(w, t)_{\Gamma_N}$, $a(w, v_D^h)$, $c(v^h, w, v_D^h)$,

$b(v_D, q)$

Matrices:

M (mass matrix) $= \int_\Omega N_i \, N_j \, d\Omega$

$$K_{ij} = \nu \int_\Omega \nabla N_i \cdot \nabla N_j$$

$$C = \int_\Omega (N_i \; \nabla N_j) \; N_i$$

$$G = -\int_\Omega \hat{N}_i \, D_j^T \, d\Omega$$

↳ being D : $\nabla \cdot \hat{N}_j$

$$f = \int_\Omega N_i \, f \, d\Omega$$

$$R = \sigma \int N_i \, N_j \, d\Omega = \sigma M$$

- Time dependent momentum equation:

$$v_t = \frac{f - [K + C(v) + R]\, v - G p}{M}$$

- Substituting v_t in theta methods:

$$\frac{v^{n+1} + v^n}{\Delta t} = \theta \left[\frac{f - [K + C(v) + R]\, v^{n+1} - G p^{n+1}}{M} - \right.$$

↳

$$- \frac{f - [K + C(v) + R] v^n - G p^n}{M} \Big] +$$

$$+ \frac{f - [K + C(v) + R] v^n - G p^n}{M}$$

$$\Rightarrow \frac{\Delta v}{\Delta t} M + \theta \left[(K + C(v) + R) \Delta v + G \Delta p \right] =$$

$$= f - [K + C(v) + R] - G p^n$$

$\Rightarrow$ System to be solved, considering $\theta = \frac{1}{2}$:

↳ adding the divergence free condition:

$$\begin{bmatrix} M + \frac{1}{2} \Delta t \, (K + C(v) + R] v^n & \frac{1}{2} \Delta t \, G \\ G^T & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta p \end{bmatrix} =$$

$$= \begin{bmatrix} \Delta t \left[f - [K + C(v) + R] v^n - G p^n \right] \\ 0 \end{bmatrix}$$

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d) To solve this implicit non-linear problem, it is proposed Newton-Raphson method:

* Non-linear system : $r(x) = 0$

* Iterative algorithm
  - Given an initial guess $\underline{x}^0$
  - Iterate until convergence is achieved:
    - Solve a linear system of equations:

$$\underline{J}(x^k)\,\Delta x^{k+1} = -r(x^k)$$

    - Update solution:

$$x^{k+1} = x^k + \Delta x^{k+1}$$

* Where the Jacobian has to be computed at each iteration:

$$[\underline{J}(x)]_{ij} = \frac{\partial r_i}{\partial x_j}(x)$$

* This implies, for the $J(x)$ calculation:

System: $r(x) = A(x)\,x - b(x)$

Jacobian is:

$$J(x) = A(x) + \underbrace{\frac{\partial A(x)}{\partial x}\,x - \frac{\partial b(x)}{\partial x}}_{\text{May be difficult to compute}}$$

e)

Both methods behave as expected.
As it is shown, for small Re, both methods converge correctly. Piccard's method has a linear convergence whereas Newton has a quadratic convergence (second order convergence).
However, for a high Reynolds, Piccard's method still has linear convergence, but Newton-Raphson diverges.
First conclusion, Newton method is less robust than Piccard although Newton gives faster convergence for lower number of Reynolds.
Newton method for high Reynolds diverges because it presents many instabilities presented in the computation of the Jacobian regarding the convection matrix of the velocity.
Remark that for really high Re, Piccard diverges as well.
The possible solutions for that is taking a better initial guess for the Newton methods. For instance, the solution given for Re = 500, where the method still converges.
Another possible solution is to implement a line-search method to the Newton to assure that the solution has a down hill direction that is going to the minimum where residual is 0.

Exercise 1

Stokes problem:

$$\begin{cases} -\nu \nabla^2 \underline{v} + \nabla p = \underline{b} & \text{in } \Omega \\ \underline{\nabla} \cdot \underline{v} = 0 & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

$Q_2 Q_1$ finite elements (quadratic)

a) This pair of elements it is called the Taylor-Hood elements ($Q_2 . Q_1$)
In order to know if this pair of elements are suitable to discretize the problem, they must fulfil the inf-sup condition (LBB condition)
Remark that as it is a Stokes problem, there are not instabilities when the Re is high due to the fact there is not convective matrix.
The inf-sup condition states that the velocity and pressure spaces have to be linked:

$$\inf_{\substack{q^h \in Q^h \\ q^h \neq 0}} \sup_{\substack{v^h \in V^h \\ v^h \neq 0}} \frac{b(v^h, q^h)}{\|v^h\|_1 \|q\|_0} \geq \beta > 0$$

So, it necessary but not sufficient condition:

$$\dim Q^h \leq \dim V^h$$

$\dim Q^h \Rightarrow Q_1$

$\dim V^h \Rightarrow Q_2$

$$\left\{ \left[Q_1 \leq Q_2 \right] \right. \checkmark$$

It is fullfilled for $Q_2 Q_1$.

↳ linear elements for pressure

↳ Quadratic elements for velocity

Moreover, the problem has a unique solution if

$\ker G^T = 0$

This means that,

$$\underbrace{(G K^{-1} G^T)}_{\text{pressure matrix is symmetric positive definite since } \ker G^T = 0} p = G K^{-1} f$$

b) As it is mentioned before, the problem fullfils the LBB and there is an unique solution up to a constant.

In conclusion, there is no need for any stabilization technique. Therefore, GLS is no needed and it will only cost an extra computational cost unnecessary.

e) Algorithm of an HDG solver for Stokes

1. Write strong form of the Stokes problem over the broken computational domain (compute for each element)
2. Impose continuity of solutions and fluxes ①
3. Write Stokes as a system of 1st order of equations on broken computational domain.

① $[\![u \otimes n]\!] = 0$ on Γ

$[\![n \cdot (\nu \nabla u - p\, I_{n_{sd}})]\!] = 0 \quad \Gamma$

3.a. Introduce the mixed variable ($\underline{L}$) which is a tensor

$$\left[\underline{L} + \sqrt{\nu}\, \nabla \underline{u} = 0\right]$$

4. Solve the local problem and introduced the hybrid variable $(\hat{u})$

$$\begin{cases} u_e = u_D & \text{on } \partial\Omega_e \cap \Gamma_D \\ \boxed{u_e = \hat{u}} & \text{on } \partial\Omega_e \setminus \Gamma_D \end{cases}$$

4.1. For Dirichlet problem, the pressure is determined up to a constant.

$$\frac{1}{|\partial\Omega_e|} \langle p_e, 1 \rangle_{\partial\Omega_e} = \boxed{\rho_e} \to \text{unknown}$$

4.2 Check:

$$\langle \hat{u} \cdot n_e, 1 \rangle_{\partial\Omega_e \setminus \Gamma_D} + \langle u_D \cdot n_e, 1 \rangle_{\partial\Omega_e \cap \Gamma_D} = 0$$

\# Solve for $L_e(\hat{u})$ and $u_e(\hat{u})$ as a function of $\hat{u}$

5. Recall the global problem, solved in the transmission conditions,

$$[\![n \cdot (-\nu L + p\, I_{n_{sd}})]\!] = 0 \quad \text{on } \Gamma$$

$$n \cdot (-\nu L + p\, I_{n_{sd}}) = -t \quad \text{on } \Gamma_N$$

$$\cancel{[\![u \otimes n]\!] = 0}$$

5.1 Define the trace of the numerical flux

$n_e \cdot (\hat{[F]} L_e^h + p_e^h I_{n_{sd}})$

After discretization the local problem $\Rightarrow$ system of equations solved by element by element with variables (L_e, u_e, p_e, ξ)

$\hookrightarrow$ lagrange multiplier

\+ When the global problem is recalled:

$$\begin{bmatrix} \hat{K} & \hat{G} \\ \hat{G}^T & 0 \end{bmatrix} \begin{Bmatrix} \hat{u} \\ \rho \end{Bmatrix} = \begin{Bmatrix} \hat{f}_u \\ \hat{f}_p \end{Bmatrix}$$

$\hookrightarrow$ unknowns: hybrid variable and ρ

c) DG

In order to fulfil the LBB condition, as it is mentioned before as for continuous Galerkin

$$\dim Q^h \leq \dim V^h$$

Therefore, it is preferable that the degree of pressure is one or two degrees inferior. However, numerical results show that using equal order spaces for velocity and pressure can also work well

The degree of the pressure is never higher than the velocity degree.

* For the computation of the size
  degree of pressure = 5.

In DG, the nodes on the edges are duplicated

7 nodes on the edges × 2 = 14 nodes in the edge

14 × 8 = 112 nodes on the edges =

↳ Velocity : 112 × 2 = 224 dofs

* For pressure
  6 nodes on edge × 2 = 12 nodes on edge
  12 × 8 = 96 dofs

* Global size
  224 dofs (vel) + 96 dofs (pressure) =

Global size = 320 dofs

In DG is computed only the nodes on the edges that are duplicated on the interface. The interior nodes do not play a rol.

d) In HDG, the size of the global problem only depends on the nodes of the edge (skeleton) and it duplicates the nodes on the vertices.

In this case, to pass the LBB condition
$\Rightarrow$ equal interpolation for interior values (L_e, u_e, p_e) and the hybrid variable $\hat{u}$

So, degree of pressure = 6

the nodes on the vertices duplicates as many times as edges end in this vertex.

+ In the local problem is computed all the dofs for each element and then gattered