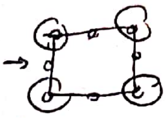
Marcos Bonget

Problem 1

$$\left\{\begin{array}{lll} -\nu\nabla^2 v + \nabla p = b & \text{in } \Omega \\ \nabla \cdot v = 0 & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{array}\right\} \text{Stokes}$$

$\boxed{Q_2Q_1 \text{ elements}} \rightarrow$ 

a) Q_2Q_1 is suitable because it is LBB stable.

For Stokes problem $\begin{pmatrix} K & G^T \\ G & 0 \end{pmatrix}\begin{pmatrix} v \\ p \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}$, if we obtain 1^{st} equation a substitute in 2^{nd} we obtain: $v = K^{-1}(f - G^T p) \rightarrow \boxed{GK^{-1}G^T}\, p = GK^{-1}f$

$GK^{-1}G^T$ is SPD if $\ker(G^T) = \{0\}$, and this only happens if spaces V or Q satisfy LBB condition or inf-sup condition. It is necessary that:

$\dim Q^h \leq \dim V^h$ with $Q = p \in L_2(\Omega)$
$V = v \in H^1(\Omega)$

[illegible]

b) Considering GLS for Stokes & Q_2Q_1, we have linear shape pressure functions and quadratic for velocity.

By adding the stabilization: $\sum_e \int_{\Omega_e} \tau \mathcal{L}(w,q) \cdot (\mathcal{L}(v,p) - F)\, d\Omega$

By adding stabilization parameters to our system:

$$\begin{bmatrix} K + \tilde{K} & G^T + \tilde{G}^T \\ -G + \tilde{G} & 0 + \tilde{L} \end{bmatrix}\begin{pmatrix} v \\ p \end{pmatrix} = \begin{pmatrix} f + \tilde{f} \\ 0 + \tilde{f}_q \end{pmatrix} \quad \text{with } K(\tau_1, \tau_2)$$

given that velocity is non-linear, $\boxed{\text{all of this}}$ additions to stabilize Stokes $\boxed{\text{matters}}$. LBB stable condition was not sufficient. τ_1, τ_2 shall be fixed (usually $\tau_2 = 0$)

c) Pressure approximation to fulfill LBB condition. degree 6. DG.

$\dim Q^h \leq \dim V^h = 6$

Pressure approximation must be of $\boxed{\text{degree} \leq 6}$

FEF EXAM

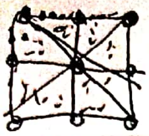
Marcos Bonguet

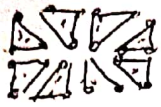
d) HDG discretization of skeletons agenten

each side 6

~~[illegible]~~

216 − 80 + 80 = 240 DOF for u





→ 6 dofs × 3 = 18 + ~~interior 0~~ = 18 dof × 12 elem } 216 DoF for $\hat{u}$

Ur (skeleton)

DOF u (primal)

being double for mixed variable q DoF = 480 DoF for q

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e) HDG algorithm: One unknown is a function on Ω and the other the trace of this function at the boundaries.

1) Global problem $\boxed{\hat{u}, z}$ → 2) local problem $\boxed{u, q}$ (*)

~~[illegible]~~ hybrid variable

First it is calculated the global system ($\hat{u}$) and z, the latter being an auxiliary set of variables.

Next thing is to calculate the local problem, calculating primal variable u and mixed variable q.

Optionally: superconvergence may be applied. The convergence rate for both u & q is $O(h^{p+1})$, but a superconvergent algorithm could have up to $O(h^{p+2})$. This postprocessed u^* is calculated solving local problem element by element.

(*) Of course global and local problem must be the first to be established, requiring the mixed variable, then setting a broken local system of a set of Ω_i subdomains and ~~[illegible]~~ setting the primal variable as ~~the~~ to be ~~equal~~ equal to the hybrid variable at the skeleton.

For the global problem, transmission conditions on the skeleton and Neumann Boundary conditions are considered.

The numerical fluxes of the HDG model are identified within the skeleton and its numerical traces must be stabilized.

Problem 2 — FEF EXAM — Marcos Borges

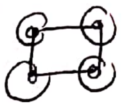
$\boxed{\text{Resistance to porosity}} \to$ N-S + σu reaction term   with $\sigma > 0$.

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \nabla^2 u + (u \cdot \nabla) u + \sigma u + \nabla p = f & \text{in } \Omega, t > 0 \\ \nabla \cdot u = 0 & \text{in } \Omega, t > 0 \\ u = 0 & \text{on } \partial\Omega, t > 0 \\ u = u_0 & \text{in } \Omega, t = 0 \end{cases}$$

u_t

new with respect Navier-Stokes

$\boxed{Q_1, Q_1 \text{ elements}}$ → not stable (LBB)



linear/linear

a) Discrete time problem CN.

Theta-method. $\boxed{\dfrac{u^{n+1} - u^n}{\Delta t} - \theta (u_t^{n+1} - u^n) = u_t^n}$ with $\theta = 1/2$ for Crank-Nicholson

↓

$$\frac{2(u^{n+1} - u^n)}{\Delta t} - u_t^{n+1} = u_t^n$$

$u_t = \nu \nabla^2 u - (u \cdot \nabla) u - \sigma u - \nabla p + f$

Resulting in: $\boxed{\dfrac{2u^{n+1}}{\Delta t} - \nu \nabla^2 u^{n+1} + (u^{n+1} \cdot \nabla) u^{n+1} + \sigma u^{n+1} + \nabla p^{n+1} = \dfrac{2u^n}{\Delta t} + f^n - \nu \nabla^2 u^n + (u^n \cdot \nabla) u^n + \sigma u^n + \nabla p^n}$

1st eqn →

2nd eqn ⟶ $\nabla \cdot u^{n+1} = 0$ (incompressibility constraint)

b) Weak form:

$$\int_\Omega w \cdot \left[\left(\frac{2}{\Delta t} + \sigma\right) u^{n+1} - \nu \nabla^2 u^{n+1} + (u^{n+1} \cdot \nabla) u^{n+1} + \nabla p^{n+1} \right] = \int_\Omega w \left[\left(\frac{2}{\Delta t} + \sigma\right) u^n - \nu \nabla^2 u^n + (u^n \cdot \nabla) u^n + \nabla p^n \right] d\Omega$$

↓ integrating by parts: $+ \int \nabla w : (\nu \nabla u^{n+1})$

↓ integrating by parts: $\int \nabla w : (\nu \nabla u^n)$

$\boxed{\text{Considering Dirichlet B.C., with } u_{\Gamma_D} = 0}$

c) $\int q \cdot (\nabla \cdot u^{n+1}) = 0$

with $\forall w \in V = L_2(\Omega)$
$\forall q \in Q := H^1(\Omega) : u = 0$ on Γ_D

FEF EXAM

for any test function — Explicit convective term

c) $\omega \cdot \left[\left(\frac{2}{\Delta t}+\sigma\right) \underbrace{\int_\Omega (\text{mat}N)^T (\text{mat}N)\, d\Omega}_{M} + \nu \underbrace{\int_\Omega [\text{grad}N]^T : [\text{grad}N]}_{K} + \int_\Omega C(v)\, d\Omega \right] u^{n+1}$

$+ \omega \left[\int_\Omega D^T \hat{N}\, d\Omega \right] p^{n+1} = \omega \cdot \left[\left(\frac{2}{\Delta t}+\sigma\right) \underbrace{\int [\text{mat}N][\text{mat}N]\, d\Omega}_{M} \cdot u^n + \nu \int_\Omega [\text{grad}N] : [\text{grad}N]\, d\Omega \cdot u^n + \int C(v^n)\, d\Omega\, v^n \right.$

$\left. + \omega \underbrace{\left[\int_\Omega D^T \hat{N} \right]}_{G^T} d\Omega\, p^n \right]$

for any test function

for any test function

$q \cdot \underbrace{\int_\Omega (\hat{N}^T D)}_{G} u^{n+1} = 0$

resulting in a matricial form:

$$(A)\quad \begin{pmatrix} \left(\frac{2}{\Delta t}+\sigma\right)M + \nu K + C(u^n) & G^T \\ G & 0 \end{pmatrix} \begin{pmatrix} u^{n+1} \\ p^{n+1} \end{pmatrix} = \begin{pmatrix} \overbrace{\left[\left(\frac{2}{\Delta t}+\sigma\right)M + \nu K + C(u^n)\right]u^n + G^T p^n}^{f^n} \\ 0 \end{pmatrix}$$

The difference is the addition of σ term with N-S

Being
$$\begin{cases} (\text{Mat}N) = \left[\begin{matrix} N_1 & 0 \\ 0 & N_1 \end{matrix} \middle| \begin{matrix} N_2 & 0 \\ 0 & N_2 \end{matrix} \middle| \cdots \begin{matrix} N_n & 0 \\ 0 & N_n \end{matrix}\right] \\ (\text{grad}N) = \left[\begin{matrix} \frac{\partial N_1}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial x} \\ \frac{\partial N_1}{\partial y} & 0 \\ 0 & \frac{\partial N_1}{\partial y} \end{matrix} \middle| \begin{matrix} \frac{\partial N_2}{\partial x} & 0 \\ 0 & \frac{\partial N_2}{\partial x} \\ \frac{\partial N_2}{\partial y} & 0 \\ 0 & \frac{\partial N_2}{\partial y} \end{matrix} \middle| \cdots \middle| \begin{matrix} \frac{\partial N_n}{\partial x} & 0 \\ 0 & \frac{\partial N_n}{\partial x} \\ \frac{\partial N_n}{\partial y} & 0 \\ 0 & \frac{\partial N_n}{\partial y} \end{matrix}\right] \\ D = (1,0,0,1)[\text{grad}N] \end{cases}$$

$$\begin{cases} M = \int_\Omega (\text{mat}N)^T (\text{mat}N)\, d\Omega \\ K = \int_\Omega (\text{grad}N)^T (\text{grad}N)\, d\Omega \\ G = \int_\Omega \hat{N}^T D\, d\Omega \end{cases}$$

see page 6 for adding FCS stabilization

d) Given that it is a non-linear system $(C(v))$, it is required a non-linear algorithm, such as Picard or Newton-Raphson. Picard is more robust and does not require the Jacobian, while N-R converges much faster, but requires calculating Jacobian and is more sensitive to initial solution u^0. (it could diverge fast).

If we choose Picard, then:

$A(x)\,x = B(x)$

with $\begin{cases} A(x) = \begin{pmatrix} \left(\frac{2}{\Delta t}+\sigma\right)M + \nu K + C(u^n) & G^T \\ G & 0 \end{pmatrix} \\ B(x) = \begin{pmatrix} f^n \\ 0 \end{pmatrix} \end{cases}$

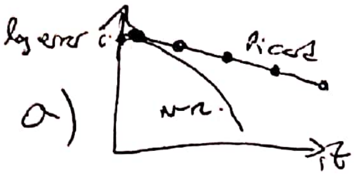
Solving linear system at each iteration, starting with $u^n = u_0$ we obtain $C(u^0)$ and we can solve linearly to obtain u^1 which is the new u^n, etc

$u^0 \to \underset{f(u^0)}{C(u^0)} \to u^1 \to \underset{f(u^1)}{C(u^1)} \to u^2 \to \cdots u^n$

Problem 2

e) $\sigma = 0$ (Pure Navier-Stokes problem)

Considering a viscous fluid around object solving (non-dimensional N-S) for $Re=10$ & $Re=1000$.

a)


b)


$Re=10$ no turbulence → laminar

$Re=1000$ high turbulence

For an unsteady ($u_t \neq 0$) viscous laminar flow ($Re \approx 1000$) the Picard regime is unstable, and may appear bifurcations on solution or complex flow patterns. The N-R case in-between this regime and the transition regime, which start to have turbulent phenomena.

For Picard's case ($Re=10$) we have a laminar flow.

b) The Jacobian may be affected by the instabilities and which may lead to divergence or chaotic behaviour on N-R

a) Both methods do converge, however N-R is much faster, as known. Picard's is an approximation of N-R.

5/6

from page 4

GLS $\boxed{\text{Stabilization}}$: for unsteady Navier-Stokes with resistance to porosity

Q1Q1 Not LBB stable. It is necessary to add stabilization parameter through GLS (Galerkin Least Squares), by imposing that the weak solution is dro solution.

$$\sum_{\Omega_e}\int_{\Omega_e}\tau\,\mathcal{L}(\underbrace{w,q}_{\text{test function}})\big(L(v,p)-F\big)\,d\Omega$$

$$\sum_{\Omega_e}\int_{\Omega_e}\tau_1\big(w_t-\cancel{\nu\nabla^2 w}+(w\cdot\nabla)w+\sigma w+\nabla q\big)\big(u_t-\cancel{\nu\nabla^2 u}+(u\cdot\nabla)u+\sigma u+\nabla p-f\big)\,d\Omega_e$$

$$+\underbrace{\sum\tau_2(\nabla\cdot w)(\nabla\cdot u)}_{2^{nd}\text{ equation}}=0$$

Important: Q1Q1 is linear for p, v. Terms with second derivative can be set to 0

$\int\tau_1 u_t^2$

$\int\tau_1 u_t(u\cdot\nabla)u$

$\int\tau_1 u_t\,\sigma u$

$\int\tau_1 u_t\,\nabla p$

$\int\tau_1 u_t(-f)$

$\tau_1\int(w\cdot\nabla)w\,u_t\ \boxed{K_1}$

$\tau_1\int(w\cdot\nabla)w\,(u\cdot\nabla)u$

$\tau_1\int(w\cdot\nabla)w\,\sigma u=\boxed{K_2}$

$\tau_1\int(w\cdot\nabla)w\,\nabla p=\boxed{G_1^T}$

$\tau_1\int(w\cdot\nabla)w(-f)=\boxed{F_{w1}}$

$\tau_1\int\sigma w\,u_t\ \boxed{K_3}$

$\tau_1\int\sigma w(u\cdot\nabla)u$

$\tau_1\int\sigma w\,\sigma u=\boxed{K_4}$

$\tau_1\int\sigma w\,\nabla p=\boxed{G_2^T}$

$\tau_1\int\sigma w(-f)=\boxed{F_{w2}}$

$\tau_1\int\nabla q\,u_t$

$\tau_1\int\nabla q\,(u\cdot\nabla)u=\boxed{G_1}$

$\tau_1\int\nabla q\,\sigma u=\boxed{G_2}$

$\tau_1\int\nabla q\,\nabla p=\boxed{L}$

$\tau_1\int\nabla q\,(-f)=\boxed{F_q}$

And $\tau_2\int\nabla w\,\nabla u=\boxed{K_5}$

$$\begin{pmatrix}(w,u)&(w,p)\\(q,u)&(q,p)\end{pmatrix}\begin{pmatrix}u\\p\end{pmatrix}=\begin{pmatrix}(w,f)\\(q,f)\end{pmatrix}$$

$$\begin{pmatrix}A&G^T\\G&0\end{pmatrix}\begin{pmatrix}u^n\\p^{n}\end{pmatrix}=\begin{pmatrix}f\\0\end{pmatrix}\quad \text{ORIGINAL}$$

Stabilized:

$$\begin{pmatrix}A+K_1+K_2+K_3+K_4+K_5&G^T+G_1^T+G_2^T\\G+G_1+G_2&0+L\end{pmatrix}\begin{pmatrix}u\\p\end{pmatrix}=\begin{pmatrix}f+F_q\\f+F_{w1}+F_{w2}\end{pmatrix}$$