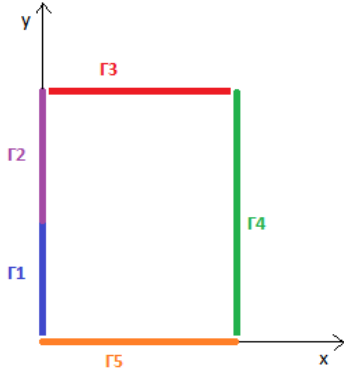


Assignment

Domain $\Omega = (0, 2) \times (0, 3)$, boundary conditions $\Gamma_1 = (0, 0) \times (0, \frac{3}{2})$, $\Gamma_2 = (0, 0) \times (\frac{3}{2}, 3)$, $\Gamma_3 = (0, 2) \times (3, 3)$, $\Gamma_4 = (2, 2) \times (0, 3)$, $\Gamma_5 = (0, 2) \times (0, 0)$.



Exercise 1

Steady convection-diffusion-reaction equation

$$\begin{cases} \mathbf{a} \cdot \nabla \rho - \nabla \cdot (\vartheta \nabla \rho) + \sigma \rho = s \\ \rho = 1 \text{ in } \Gamma_2, \rho = 0 \text{ in } \Gamma_4 \end{cases}$$

$$a) \int_{\Omega} w(\mathbf{a} \cdot \nabla \rho) d\Omega - \int_{\Omega} w \nabla \cdot (\vartheta \nabla \rho) d\Omega + \int_{\Omega} w \sigma \rho d\Omega = \int_{\Omega} w s d\Omega$$

Integrating by parts:

$$\int_{\Omega} w(\mathbf{a} \cdot \nabla \rho) d\Omega + \int_{\Omega} \nabla w \cdot (\vartheta \nabla \rho) d\Omega + \int_{\Omega} w \sigma \rho d\Omega = \int_{\Omega} w s d\Omega + \int_{\Gamma_N} w h d\Gamma$$

Let us rewrite as $a(w, \rho) + c(\mathbf{a}; w, \rho) + (w, \sigma \rho) = (w, s) + (w, h)_{\Gamma_N}$ (1) where

$$a(w, \rho) = \int_{\Omega} \nabla w \cdot (\vartheta \nabla \rho) d\Omega, c(\mathbf{a}; w, \rho) = \int_{\Omega} w(\mathbf{a} \cdot \nabla \rho) d\Omega, (w, \sigma \rho) = \int_{\Omega} w \sigma \rho d\Omega,$$

$$(w, s) = \int_{\Omega} w s d\Omega, (w, h)_{\Gamma_N} = \int_{\Gamma_N} w h d\Gamma.$$

Using approximation $\rho(x) \approx \rho^h(x) = \sum_j \rho_j N_j(x)$, receive system of equations with $w \approx \sum_i N_i(x)$:

$$\left(a(N_i, N_j)_{\Omega^e} + c(\mathbf{a}; N_i, N_j)_{\Omega^e} + \sigma(N_i, N_j)_{\Omega^e} \right) \rho_j = (N_i, s)_{\Omega^e} + (N_i, h)_{d\Omega^e \cap \Gamma_N}$$

The final system of equations takes the view: $(C+K+M) \rho = f$ where:

$$C - \text{convection matrix which consists of } C_{ij}^e = \int_{\Omega^e} N_i(\mathbf{a} \cdot \nabla N_j) d\Omega;$$

$$K - \text{diffusion matrix which consists of } K_{ij}^e = \int_{\Omega^e} \nabla N_i \cdot (\vartheta \nabla N_j) d\Omega;$$

M – mass matrix which consist of $M_{ij}^e = \sigma \int_{\Omega^e} N_i N_j d\Omega$;

f – right hand side vector which consist of elements $f_i^e = \int_{\Omega^e} N_i s d\Omega + \int_{d\Omega^e \cap \Gamma_N} N_i h d\Gamma$.

After discretization in 1D receive:

$$a \frac{\rho_{i+1} - \rho_{i-1}}{2h} - \vartheta \frac{\rho_{i+1} - 2\rho_i + \rho_{i-1}}{h^2} + \sigma \rho_i = s_i$$

In 2D:

$$K_{ij}^e + C_{ij}^e + M_{ij}^e = \sum_{ij} \vartheta \left(\frac{\partial N_i}{\partial x}(\mathbf{z}_{ig}) \frac{\partial N_j}{\partial x}(\mathbf{z}_{ig}) + \frac{\partial N_i}{\partial y}(\mathbf{z}_{ig}) \frac{\partial N_j}{\partial y}(\mathbf{z}_{ig}) \right) |\mathbf{J}(\mathbf{z}_{ig})| w_{ig}$$

$$+ \sum_{ij} N_i(\mathbf{z}_{ig}) \left(a_x \frac{\partial N_j}{\partial x}(\mathbf{z}_{ig}) + a_y \frac{\partial N_j}{\partial y}(\mathbf{z}_{ig}) \right) |\mathbf{J}(\mathbf{z}_{ig})| w_{ig} + \sum_{ij} \sigma N_i(\mathbf{z}_{ig}) N_j(\mathbf{z}_{ig}) |\mathbf{J}(\mathbf{z}_{ig})| w_{ig}$$

$$f_i^e = \sum_{ij} N_i(\mathbf{z}_{ig}) s |\mathbf{J}(\mathbf{z}_{ig})| w_{ig}.$$

b) Let us consider the problem with following conditions and spatial discretization $h=0.2$:

1. $a = 1, \vartheta = 0.001, \sigma = 0.001, s = 0$
2. $a = 0.001, \vartheta = 0.001, \sigma = 1, s = 0$
3. $a = 1, \vartheta = 0.001, \sigma = 0, s = 1$
4. $a = 1, \vartheta = 0.001, \sigma = 1, s = 0$

Convective term is in the direction X, thus, the velocity vector is taken as $a \cdot \left[\frac{\pi}{6}, 0 \right]$.

Solving the equation with Galerkin method, we will see that for 1, 3 and 4 examples Galerkin solution provides oscillations while for the 2 example solution is close to exact one.

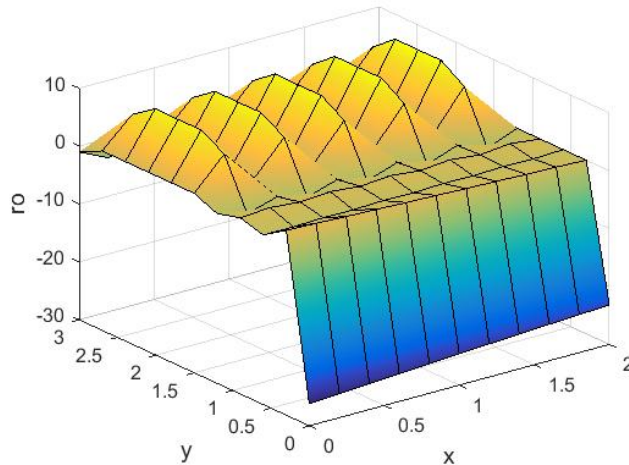


Fig. 1: Galerkin method, $a=1, \vartheta=0.001, \sigma=0.001, s=0$

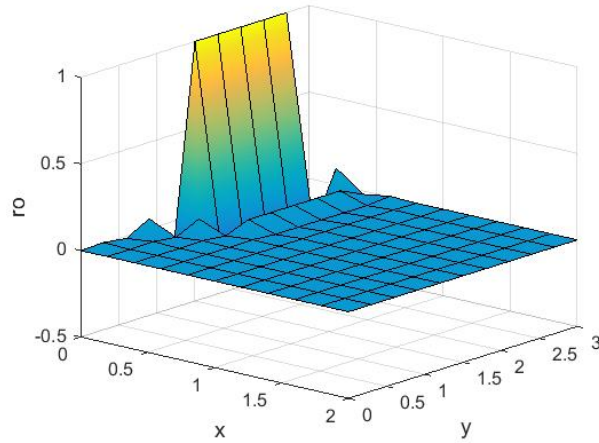


Fig. 2: Galerkin method, $a=0.001$, $\vartheta=0.001$, $\sigma=1$, $s=0$

This behavior is observed as in 1, 3 and 4 cases the Peclet number = 100 whilst for the second case $Pe=0.1$ and Galerkin method provides stable solution only if $Pe < 5$. To improve the solution for these three cases, it is necessary to reduce a spatial discretization parameter. As $Pe = \frac{ah}{2\vartheta} < 5 \rightarrow h < \frac{10\vartheta}{a}$ which means that for 1, 3 and 4 cases spatial discretization parameter must satisfy to inequality $h < 0.01$. As it can be seen from the figure 3, the Galerkin solution for $h = \frac{1}{101}$ is close to exact solution just with slight fluctuations. In this case $Pe=4.9505$.

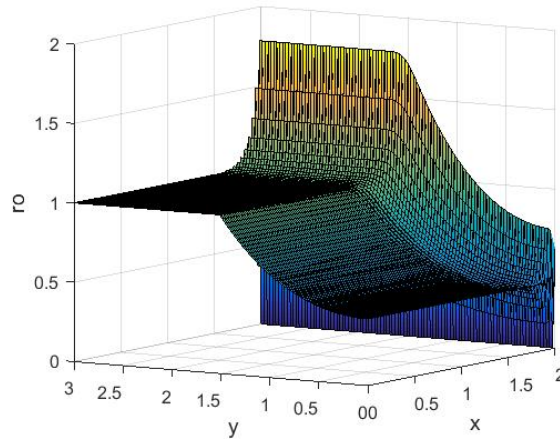


Fig. 3: Galerkin method, $a=1$, $\vartheta=0.001$, $\sigma=0.001$, $s=0$, $h=1/101$

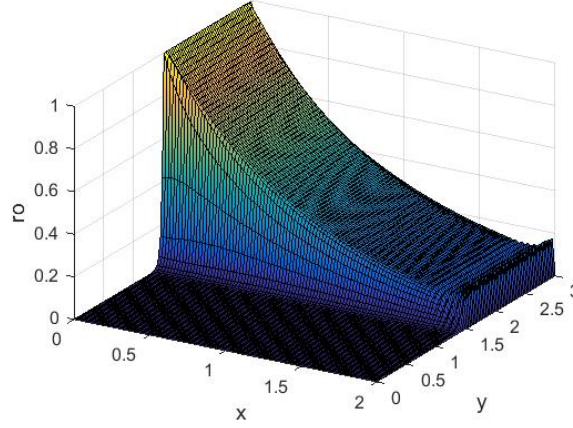


Fig. 4: Galerkin method, $a=1, \vartheta=0.001, \sigma=1, s=0, h=1/101$

- c) Now let us consider stabilization techniques, SUPG and GLS methods. These methods are unconditionally stable. Stabilization parameter is counted as $\tau = \frac{h}{2a} \left(1 + \frac{9}{Pe^2} + \left(\frac{h}{2a} \sigma \right)^2 \right)^{-1/2}$ (2).

As unconditionally stable, these methods provide solutions which are close to exact one for every from four cases. However, these solutions can have some fluctuations which are caused by Galerkin term σw from $\mathcal{P}(w) = \mathcal{L}(w) = a \cdot \nabla w - \nabla \cdot (\vartheta \nabla w) + \sigma w$.

The qualitative influence of each term of stabilization term $\mathcal{P}(w) = \mathcal{L}(w) = a \cdot \nabla w - \nabla \cdot (\vartheta \nabla w) + \sigma w$ for GLS method where $a \cdot \nabla w$ corresponds to SUPG method and σw is a Galerkin weighting. For linear elements and a constant positive reaction, GLS and SUPG with Galerkin weighted $1 + \sigma \tau$ times more. This means that the instabilities introduced by Galerkin are little more amplified in GLS to compare with SUPG. This instability is overcome in SGS method. As $\mathcal{P}(w) = -\mathcal{L}^*(w) = a \cdot \nabla w + \nabla \cdot (\vartheta \nabla w) - \sigma w$, in this case Galerkin term is weighted by $1 - \sigma \tau$ and thus has less influence than SUPG.

The third set of parameters was chosen $a = 1, \vartheta = 0.001, \sigma = 0, s = 1$. In this case $Pe = 100$ if $h=0.2$ which means that Galerkin method provides unstable solution with oscillations. Also, $\sigma = 0$ which means that for SUPG and GLS there are no fluctuations provided by Galerkin term as it is equal to zero. Thus, GLS and SUPG methods provide solution with high accuracy on given set of parameters.

To use stabilization techniques the equation can be represented in the following view:

$$\mathcal{R}(\rho) = a \cdot \nabla \rho - \nabla \cdot (\vartheta \nabla \rho) + \sigma \rho - s = \mathcal{L}(\rho) - s$$

Where $\mathcal{R}(\rho)$ is residual. Then the equation (1) takes the view:

$$a(w, \rho) + c(\mathbf{a}; w, \rho) + (w, \sigma \rho) + \sum_e \int_{\Omega^e} \mathcal{P}(w) \tau \mathcal{R}(\rho) d\Omega = (w, s) + (w, h)_{\Gamma_N} \quad (3)$$

SUPG method

This stabilization technique is defined by taking $\mathcal{P}(w) = a \cdot \nabla w$.

The system (3) takes the view:

$$\begin{aligned}
& a(w^h, \rho^h) + c(\mathbf{a}; w^h, \rho^h) + (w^h, \sigma \rho^h) + \sum_e \int_{\Omega^e} (\mathbf{a} \cdot \nabla w^h) \tau [\mathbf{a} \cdot \nabla \rho^h - \nabla \cdot (\vartheta \nabla \rho^h) + \sigma \rho^h] d\Omega \\
& = (w^h, s) + \sum_e \int_{\Omega^e} (\mathbf{a} \cdot \nabla w^h) \tau s d\Omega
\end{aligned}$$

Here $\rho(x) \approx \rho^h(x) = \sum_j \rho_j N_j(x)$, $w(x) \approx w^h(x) = \sum_i N_i(x)$, τ is a stabilization parameter which can be defined by different ways. In the current work the definition (2) is used. Receive the following system of equations:

$$\begin{aligned}
& \left(a(N_i, N_j)_{\Omega^e} + c(\mathbf{a}; N_i, N_j)_{\Omega^e} + \sigma(N_i, N_j)_{\Omega^e} + \int_{\Omega^e} (\mathbf{a} \cdot \nabla N_i) \tau [\mathbf{a} \cdot \nabla N_j - \nabla \cdot (\vartheta \nabla N_j) + \sigma N_j] d\Omega \right) \rho_j \\
& = (N_i, s)_{\Omega^e} + \int_{\Omega^e} (\mathbf{a} \cdot \nabla N_i) \tau s d\Omega
\end{aligned}$$

The derivatives in physical coordinates (x, y) $\nabla_{xy} = \mathbf{J}^{-1} \nabla_{\xi\eta}$. The modified weights $dx dy = |\mathbf{J}| d\xi d\eta$. The vector N_{ig} has the value of the shape functions at the current Gauss point \mathbf{z}_{ig} . For a bilinear quadrilateral element is used, $N_{ig} = [N_1(\mathbf{z}_{ig}), N_2(\mathbf{z}_{ig}), N_3(\mathbf{z}_{ig}), N_4(\mathbf{z}_{ig})]$. Then, $N_{ig}' N_{ig}$ is a 4x4 matrix.

This system can be represented as $\mathbf{K} \rho = \mathbf{f}$ where where matrix \mathbf{K} consists of elements

$$\begin{aligned}
K_{ij}^e & = a(N_i, N_j)_{\Omega^e} + c(\mathbf{a}; N_i, N_j)_{\Omega^e} + \sigma(N_i, N_j)_{\Omega^e} + \int_{\Omega^e} (\mathbf{a} \cdot \nabla N_i) \tau [\mathbf{a} \cdot \nabla N_j - \nabla \cdot (\vartheta \nabla N_j) + \sigma N_j] d\Omega \\
& = \int_{\Omega^e} \vartheta \left(\frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} \right) d\Omega + \int_{\Omega^e} N_i \left(a_x \frac{\partial N_j}{\partial x} + a_y \frac{\partial N_j}{\partial y} \right) d\Omega + \int_{\Omega^e} \sigma N_i N_j d\Omega \\
& + \int_{\Omega^e} \left(a_x \frac{\partial N_i}{\partial x} + a_y \frac{\partial N_i}{\partial y} \right) \tau \left[a_x \frac{\partial N_j}{\partial x} + a_y \frac{\partial N_j}{\partial y} - \vartheta \left(\frac{\partial^2 N_j}{\partial x^2} + \frac{\partial^2 N_j}{\partial y^2} \right) + \sigma N_j \right] d\Omega \\
& \approx \sum_{ij} \vartheta \left(\frac{\partial N_i}{\partial x}(\mathbf{z}_{ig}) \frac{\partial N_j}{\partial x}(\mathbf{z}_{ig}) + \frac{\partial N_i}{\partial y}(\mathbf{z}_{ig}) \frac{\partial N_j}{\partial y}(\mathbf{z}_{ig}) \right) |\mathbf{J}(\mathbf{z}_{ig})| w_{ig} \\
& + \sum_{ij} N_i(\mathbf{z}_{ig}) \left(a_x \frac{\partial N_j}{\partial x}(\mathbf{z}_{ig}) + a_y \frac{\partial N_j}{\partial y}(\mathbf{z}_{ig}) \right) |\mathbf{J}(\mathbf{z}_{ig})| w_{ig} + \sum_{ij} \sigma N_i(\mathbf{z}_{ig}) N_j(\mathbf{z}_{ig}) |\mathbf{J}(\mathbf{z}_{ig})| w_{ig} \\
& + \sum_{ij} \left(a_x \frac{\partial N_i}{\partial x}(\mathbf{z}_{ig}) + a_y \frac{\partial N_i}{\partial y}(\mathbf{z}_{ig}) \right) \tau \left(a_x \frac{\partial N_j}{\partial x}(\mathbf{z}_{ig}) + a_y \frac{\partial N_j}{\partial y}(\mathbf{z}_{ig}) \right. \\
& \left. + \vartheta \left(\frac{\partial^2 N_j}{\partial x^2}(\mathbf{z}_{ig}) + \frac{\partial^2 N_j}{\partial y^2}(\mathbf{z}_{ig}) \right) + \sigma N_j(\mathbf{z}_{ig}) \right) |\mathbf{J}(\mathbf{z}_{ig})| w_{ig}
\end{aligned}$$

vector \mathbf{f} consists of elements

$$\begin{aligned}
f_i^e & = (N_i, s)_{\Omega^e} + \int_{\Omega^e} (\mathbf{a} \cdot \nabla N_i) \tau s d\Omega \\
& \approx \sum_{ij} N_i(\mathbf{z}_{ig}) s |\mathbf{J}(\mathbf{z}_{ig})| w_{ig} + \sum_{ij} \left(a_x \frac{\partial N_i}{\partial x}(\mathbf{z}_{ig}) + a_y \frac{\partial N_i}{\partial y}(\mathbf{z}_{ig}) \right) \tau s |\mathbf{J}(\mathbf{z}_{ig})| w_{ig}
\end{aligned}$$

The following changes of the code to compare with Galerkin method were made:

- 1) Matrix \mathbf{K} is defined as follow:

$$K_e = K_e + (\nu^*(N_x' * N_x + N_y' * N_y) + N_{ig}' * (a_x * N_x + a_y * N_y) + \sigma * (N_{ig}' * N_{ig}) + \tau * (a_x * N_x + a_y * N_y)' * (a_x * N_x + a_y * N_y + \sigma * N_{ig})) * d\text{volu};$$

$$\text{Where } \nu^*(N_x' * N_x + N_y' * N_y) = \sum_{ij} \vartheta \left(\frac{\partial N_i}{\partial x}(\mathbf{z}_{ig}) \frac{\partial N_j}{\partial x}(\mathbf{z}_{ig}) + \frac{\partial N_i}{\partial y}(\mathbf{z}_{ig}) \frac{\partial N_j}{\partial y}(\mathbf{z}_{ig}) \right),$$

$$N_{ig}' * (a_x * N_x + a_y * N_y) = \sum_{ij} N_i(\mathbf{z}_{ig}) \left(a_x \frac{\partial N_j}{\partial x}(\mathbf{z}_{ig}) + a_y \frac{\partial N_j}{\partial y}(\mathbf{z}_{ig}) \right),$$

$$\sigma * (N_{ig}' * N_{ig}) = \sum_{ij} \sigma N_i(\mathbf{z}_{ig}) N_j(\mathbf{z}_{ig}),$$

$$\tau * (a_x * N_x + a_y * N_y)' * (a_x * N_x + a_y * N_y + \sigma * N_{ig}) = \sum_{ij} \left(a_x \frac{\partial N_i}{\partial x}(\mathbf{z}_{ig}) + \right.$$

$$\left. a_y \frac{\partial N_i}{\partial y}(\mathbf{z}_{ig}) \right) \tau a_x \frac{\partial N_j}{\partial x}(\mathbf{z}_{ig}) + a_y \frac{\partial N_j}{\partial y}(\mathbf{z}_{ig}) + \vartheta \frac{\partial^2 N_j}{\partial x^2}(\mathbf{z}_{ig}) + \frac{\partial^2 N_j}{\partial y^2}(\mathbf{z}_{ig}) + \sigma N_j(\mathbf{z}_{ig}), \quad d\text{volu} = \text{wgp}(\text{ig}) * \det(\text{Jacob}) = |\mathbf{J}(\mathbf{z}_{ig})| w_{ig}$$

2) Vector f is defined as follow:

$$f_e = f_e + (N_{ig} + \tau * (a_x * N_x + a_y * N_y))' * (s * d\text{volu}). \quad \text{Where } N_{ig} = \sum_{ij} N_i(\mathbf{z}_{ig}),$$

$$\tau * (a_x * N_x + a_y * N_y) = \sum_{ij} \left(a_x \frac{\partial N_i}{\partial x}(\mathbf{z}_{ig}) + a_y \frac{\partial N_i}{\partial y}(\mathbf{z}_{ig}) \right) \tau, \quad s * d\text{volu} = \tau s |\mathbf{J}(\mathbf{z}_{ig})| w_{ig}$$

GLS method

For GLS method $\mathcal{P}(w) = \mathcal{L}(w) = \mathbf{a} \cdot \nabla w - \nabla \cdot (\vartheta \nabla w) + \sigma w$.

Analogically to the previous method, the system (3) takes the view:

$$\mathbf{a}(w^h, \rho^h) + c(\mathbf{a}; w^h, \rho^h) + (w^h, \sigma \rho^h)$$

$$+ \int_{\Omega^e} (\mathbf{a} \cdot \nabla w^h - \nabla \cdot (\vartheta \nabla w^h) + \sigma w^h) \tau [\mathbf{a} \cdot \nabla \rho^h - \nabla \cdot (\vartheta \nabla \rho^h) + \sigma \rho^h] d\Omega$$

$$= (w^h, s) + \int_{\Omega^e} (\mathbf{a} \cdot \nabla w^h - \nabla \cdot (\vartheta \nabla w^h) + \sigma w^h) \tau s d\Omega$$

$$\left(\mathbf{a}(N_i, N_j)_{\Omega^e} + c(\mathbf{a}; N_i, N_j)_{\Omega^e} + \sigma(N_i, N_j)_{\Omega^e} \right.$$

$$\left. + \int_{\Omega^e} (\mathbf{a} \cdot \nabla N_i - \nabla \cdot (\vartheta \nabla N_i) + \sigma N_i) \tau [\mathbf{a} \cdot \nabla N_j - \nabla \cdot (\vartheta \nabla N_j) + \sigma N_j] d\Omega \right) \rho_j$$

$$= (N_i, s)_{\Omega^e} + \int_{\Omega^e} (\mathbf{a} \cdot \nabla N_i - \nabla \cdot (\vartheta \nabla N_i) + \sigma N_i) \tau s d\Omega$$

The system can be rewritten as $\mathbf{K} \rho = f$ where matrix \mathbf{K} consists of elements

$$\begin{aligned}
K_{ij}^e &= a(N_i, N_j)_{\Omega^e} + c(\mathbf{a}; N_i, N_j)_{\Omega^e} + \sigma(N_i, N_j)_{\Omega^e} \\
&+ \int_{\Omega^e} (\mathbf{a} \cdot \nabla N_i - \nabla \cdot (\vartheta \nabla N_i) + \sigma N_i) \tau [\mathbf{a} \cdot \nabla N_j - \nabla \cdot (\vartheta \nabla N_j) + \sigma N_j] d\Omega \\
&= \int_{\Omega^e} \vartheta \left(\frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} \right) d\Omega + \int_{\Omega^e} N_i \left(a_x \frac{\partial N_j}{\partial x} + a_y \frac{\partial N_j}{\partial y} \right) d\Omega + \int_{\Omega^e} \sigma N_i N_j d\Omega \\
&+ \int_{\Omega^e} \left(a_x \frac{\partial N_i}{\partial x} + a_y \frac{\partial N_i}{\partial y} - \vartheta \left(\frac{\partial^2 N_i}{\partial x^2} + \frac{\partial^2 N_i}{\partial y^2} \right) + \sigma N_i \right) \tau \left[a_x \frac{\partial N_j}{\partial x} + a_y \frac{\partial N_j}{\partial y} - \vartheta \left(\frac{\partial^2 N_j}{\partial x^2} + \frac{\partial^2 N_j}{\partial y^2} \right) \right. \\
&\left. + \sigma N_j \right] d\Omega \\
&\approx \sum_{ij} \vartheta \left(\frac{\partial N_i}{\partial x}(\mathbf{z}_{ig}) \frac{\partial N_j}{\partial x}(\mathbf{z}_{ig}) + \frac{\partial N_i}{\partial y}(\mathbf{z}_{ig}) \frac{\partial N_j}{\partial y}(\mathbf{z}_{ig}) \right) |\mathbf{J}(\mathbf{z}_{ig})| w_{ig} \\
&+ \sum_{ij} N_i(\mathbf{z}_{ig}) \left(a_x \frac{\partial N_j}{\partial x}(\mathbf{z}_{ig}) + a_y \frac{\partial N_j}{\partial y}(\mathbf{z}_{ig}) \right) |\mathbf{J}(\mathbf{z}_{ig})| w_{ig} + \sum_{ij} \sigma N_i(\mathbf{z}_{ig}) N_j(\mathbf{z}_{ig}) |\mathbf{J}(\mathbf{z}_{ig})| w_{ig} \\
&+ \sum_{ij} \left(a_x \frac{\partial N_i}{\partial x}(\mathbf{z}_{ig}) + a_y \frac{\partial N_i}{\partial y}(\mathbf{z}_{ig}) + \vartheta \left(\frac{\partial^2 N_i}{\partial x^2}(\mathbf{z}_{ig}) + \frac{\partial^2 N_i}{\partial y^2}(\mathbf{z}_{ig}) \right) \right. \\
&\left. + \sigma N_i(\mathbf{z}_{ig}) \right) \tau \left(a_x \frac{\partial N_j}{\partial x}(\mathbf{z}_{ig}) + a_y \frac{\partial N_j}{\partial y}(\mathbf{z}_{ig}) + \vartheta \left(\frac{\partial^2 N_j}{\partial x^2}(\mathbf{z}_{ig}) + \frac{\partial^2 N_j}{\partial y^2}(\mathbf{z}_{ig}) \right) \right. \\
&\left. + \sigma N_j(\mathbf{z}_{ig}) \right) |\mathbf{J}(\mathbf{z}_{ig})| w_{ig}
\end{aligned}$$

vector f consists of elements

$$\begin{aligned}
f_i^e &= (N_i, s)_{\Omega^e} + \int_{\Omega^e} (\mathbf{a} \cdot \nabla N_i - \nabla \cdot (\vartheta \nabla N_i) + \sigma N_i) \tau s d\Omega \\
&\approx \sum_{ij} N_i(\mathbf{z}_{ig}) s |\mathbf{J}(\mathbf{z}_{ig})| w_{ig} \\
&+ \sum_{ij} \left(a_x \frac{\partial N_i}{\partial x}(\mathbf{z}_{ig}) + a_y \frac{\partial N_i}{\partial y}(\mathbf{z}_{ig}) + \vartheta \left(\frac{\partial^2 N_i}{\partial x^2}(\mathbf{z}_{ig}) + \frac{\partial^2 N_i}{\partial y^2}(\mathbf{z}_{ig}) \right) \right. \\
&\left. + \sigma N_i(\mathbf{z}_{ig}) \right) \tau s |\mathbf{J}(\mathbf{z}_{ig})| w_{ig}
\end{aligned}$$

1) Matrix K is defined as follow:

$$\begin{aligned}
K_e &= K_e + (\text{nu}^*(N_x' * N_x + N_y' * N_y) + N_ig' * (a_x * N_x + a_y * N_y) + \\
&\text{sigma} * (N_ig' * N_ig) + \\
&\text{tau} * (a_x * N_x + a_y * N_y + \text{sigma} * N_ig) ' * (a_x * N_x + a_y * N_y + \text{sigma} * N_ig)) * \text{dvolu};
\end{aligned}$$

$$\text{Where } \text{nu}^*(N_x' * N_x + N_y' * N_y) = \sum_{ij} \vartheta \left(\frac{\partial N_i}{\partial x}(\mathbf{z}_{ig}) \frac{\partial N_j}{\partial x}(\mathbf{z}_{ig}) + \frac{\partial N_i}{\partial y}(\mathbf{z}_{ig}) \frac{\partial N_j}{\partial y}(\mathbf{z}_{ig}) \right),$$

$$N_ig' * (a_x * N_x + a_y * N_y) = \sum_{ij} N_i(\mathbf{z}_{ig}) \left(a_x \frac{\partial N_j}{\partial x}(\mathbf{z}_{ig}) + a_y \frac{\partial N_j}{\partial y}(\mathbf{z}_{ig}) \right),$$

$$\text{sigma} * (N_ig' * N_ig) = \sum_{ij} \sigma N_i(\mathbf{z}_{ig}) N_j(\mathbf{z}_{ig}),$$

$$\text{tau} * (a_x * N_x + a_y * N_y + \text{sigma} * N_ig) ' * (a_x * N_x + a_y * N_y + \text{sigma} * N_ig)$$

$$= \sum_{ij} \left(a_x \frac{\partial N_i}{\partial x}(\mathbf{z}_{ig}) + a_y \frac{\partial N_i}{\partial y}(\mathbf{z}_{ig}) + \vartheta \left(\frac{\partial^2 N_i}{\partial x^2}(\mathbf{z}_{ig}) + \frac{\partial^2 N_i}{\partial y^2}(\mathbf{z}_{ig}) \right) + \sigma N_i(\mathbf{z}_{ig}) \right) \tau \left(a_x \frac{\partial N_j}{\partial x}(\mathbf{z}_{ig}) + a_y \frac{\partial N_j}{\partial y}(\mathbf{z}_{ig}) + \vartheta \frac{\partial^2 N_j}{\partial x^2}(\mathbf{z}_{ig}) + \vartheta \frac{\partial^2 N_j}{\partial y^2}(\mathbf{z}_{ig}) + \sigma N_j(\mathbf{z}_{ig}) \right)$$

2) Vector f is defined as follow:

$$f_e = f_e + (N_{ig} + \tau \cdot (a_x \cdot N_x + a_y \cdot N_y + \sigma \cdot N_{ig})) \cdot (s \cdot dvolu)$$

Where $N_{ig} = \sum_{ij} N_i(\mathbf{z}_{ig})$, $N_{ig} + \tau \cdot (a_x \cdot N_x + a_y \cdot N_y + \sigma \cdot N_{ig}) = \sum_{ij} \left(a_x \frac{\partial N_i}{\partial x}(\mathbf{z}_{ig}) + a_y \frac{\partial N_i}{\partial y}(\mathbf{z}_{ig}) + \vartheta \frac{\partial^2 N_i}{\partial x^2}(\mathbf{z}_{ig}) + \vartheta \frac{\partial^2 N_i}{\partial y^2}(\mathbf{z}_{ig}) + \sigma N_i(\mathbf{z}_{ig}) \right) \tau$, $s \cdot dvolu = \tau \cdot \mathbf{z}_{ig} \cdot \mathbf{z}_{ig}$

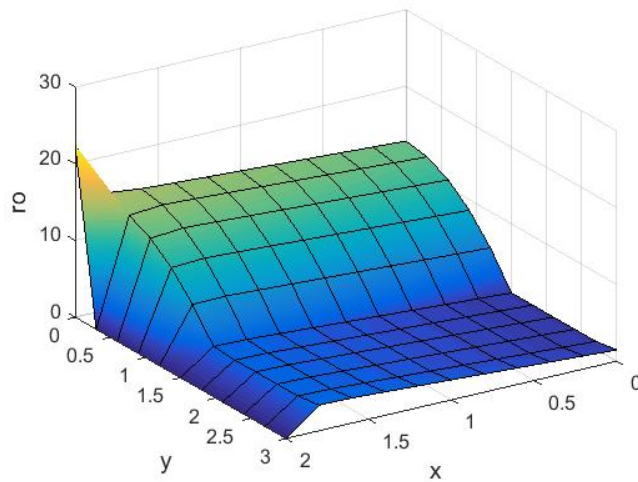


Fig. 5: SUPG, $a=1$, $\vartheta=0.001$, $\sigma=0$, $s=1$, $\tau = 0.12496$

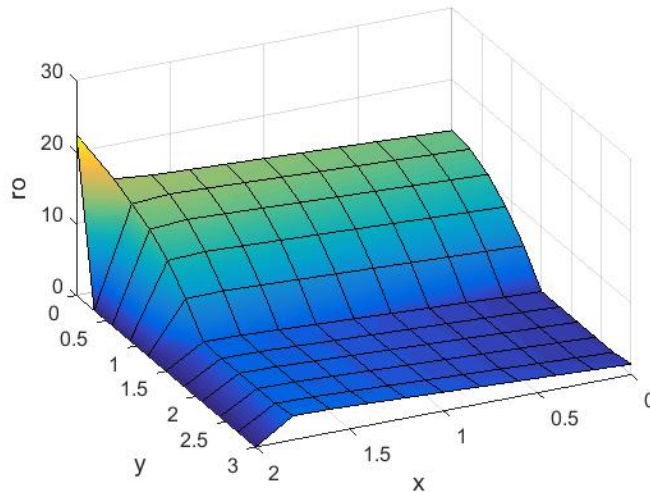


Fig. 6: GLS, $a=1$, $\vartheta=0.001$, $\sigma=0$, $s=1$, $\tau = 0.12496$

As it clearly seen from figures 7 and 8, GLS and SUPG solutions are completely similar. This happened as $\sigma = 0$ which means that Galerkin term is equal to zero and there are no fluctuations caused by it.

Now let us consider the problem with the previous set of parameters and new boundary conditions $\rho = 2$ in Γ_2 , $\rho = 1$ in Γ_4 .

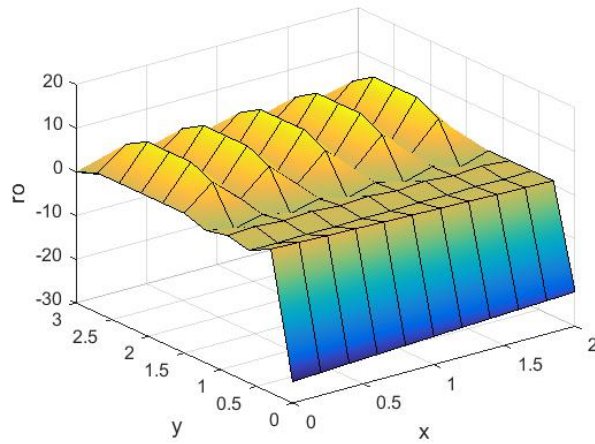


Fig. 7: Galerkin, $a=1$, $\vartheta=0.001$, $\sigma=1$, $s=0$

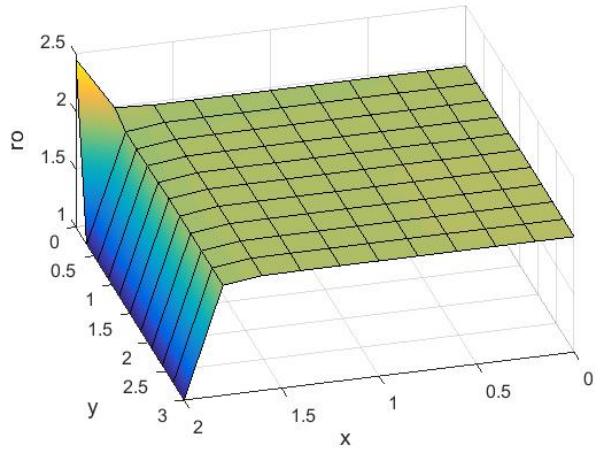


Fig. 8: SUPG, $a=1$, $\vartheta=0.001$, $\sigma=1$, $s=0$, $\tau = 0.12496$

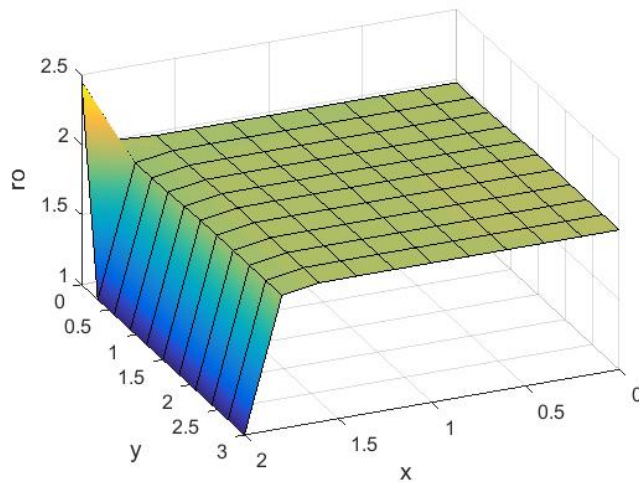


Fig. 9: GLS, $a=1$, $\vartheta=0.001$, $\sigma=1$, $s=0$, $\tau = 0.12496$

As it seen from the figures 9, 10 and 11, with new boundary conditions methods reflect the similar behavior. Galerkin solution provide whild oscillations as $Pe = 100$. SUPG and GLS provide solutions with high accuracy with some fluctuations close to boundary which are caused by Galerkin term.

Now let us consider the case when Neumann boundary conditions are imposed in Γ_4 .

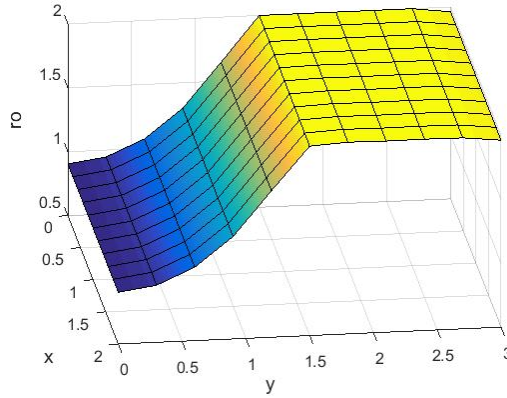


Fig. 10: Galerkin, $a=1, \vartheta=0.001, \sigma=1, s=0$

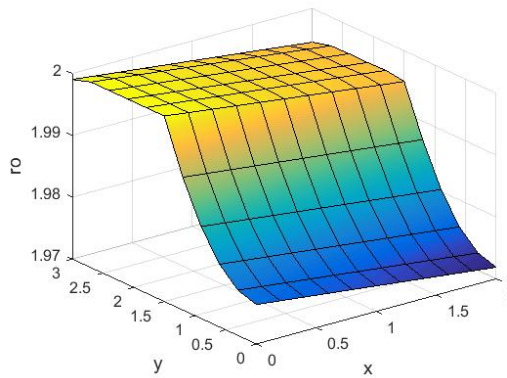


Fig. 11: SUPG, $a=1, \vartheta=0.001, \sigma=1, s=0, \tau = 0.12496$

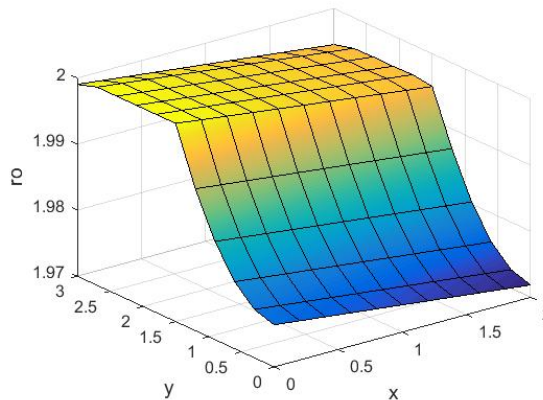


Fig. 12: GLS, $a=1, \vartheta=0.001, \sigma=1, s=0, \tau = 0.12496$

From figures 10, 11 and 12 it seen that as Peclet number is still high ($Pe=100$), Galerkin solution has noticeable deviation of SUPG and GLS solutions which are similar and have big accuracy.

Exercise 2

Let us consider transient convective-diffusion-reaction problem:

$$\begin{cases} \rho_t + \mathbf{a} \cdot \nabla \rho - \nabla \cdot (\vartheta \nabla \rho) + \sigma \rho = s \\ \rho(x, 0) = x(2 - x) \\ \rho = 1 \text{ in } \Gamma_2, \quad \rho = 0 \text{ in } \Gamma_4 \end{cases}$$

With the vector $\mathbf{a}(-x, -y)$.

a) For spatial discretization of the problem was used SUPG method as an unconditionally stable method.

As it was developed in the previous exercise, the spatial part is discretized as following:

For time discretization Crank-Nicolson and Pade methods were chosen.

1. Crank-Nicolson method

Crank-Nicolson is a θ -method with $\theta = 1/2$. For time $t^{n+1} = t^n + \Delta t$:

$$\frac{\Delta \rho}{\Delta t} = \frac{\rho(t^{n+1}) - \rho(t^n)}{\Delta t} = \theta \rho_t(t^{n+1}) + (1 - \theta) \rho_t(t^n) + O\left(\left(\frac{1}{2} - \theta\right) \Delta t, \Delta t^2\right)$$

Neglecting last term, receive:

$$\frac{\Delta \rho}{\Delta t} - \frac{1}{2} \Delta \rho_t = \rho_t^n, \quad \text{where } \Delta \rho = \rho^{n+1} - \rho^n$$

From the convective-diffusion-reaction equation $\rho_t = -\mathbf{a} \cdot \nabla \rho + \nabla \cdot (\vartheta \nabla \rho) - \sigma \rho + s$, putting it to the previous equation:

$$\begin{aligned} \frac{\Delta \rho}{\Delta t} + \frac{1}{2} (\mathbf{a} \cdot \nabla) \Delta \rho - \frac{1}{2} \nabla \cdot (\vartheta \nabla) \Delta \rho + \frac{1}{2} \sigma \Delta \rho - \frac{1}{2} (s^{n+1} - s^n) &= -\mathbf{a} \cdot \nabla \rho^n + \nabla \cdot (\vartheta \nabla \rho^n) - \sigma \rho^n + s^n \rightarrow \\ \frac{\Delta \rho}{\Delta t} + \frac{1}{2} ((\mathbf{a} \cdot \nabla) - \nabla \cdot (\vartheta \nabla) + \sigma) \Delta \rho &= \frac{1}{2} (s^{n+1} + s^n) + (-\mathbf{a} \cdot \nabla + \nabla \cdot (\vartheta \nabla) - \sigma) \rho^n \end{aligned}$$

Now let us apply spatial approximation with SUPG method. Receiving a weak form:

$$\left(w, \frac{\Delta \rho}{\Delta t}\right) + \frac{1}{2} (w, ((\mathbf{a} \cdot \nabla) - \nabla \cdot (\vartheta \nabla) + \sigma) \Delta \rho) = \frac{1}{2} (w, (s^{n+1} + s^n)) + (w, (-\mathbf{a} \cdot \nabla + \nabla \cdot (\vartheta \nabla) - \sigma) \rho^n)$$

Integrating by parts and introducing the optimization term:

$$\begin{aligned} \left(w, \frac{\Delta \rho}{\Delta t}\right) + \frac{1}{2} (a(w, \Delta \rho) + c(\mathbf{a}; w, \Delta \rho) + (w, \sigma \Delta \rho) + \sum_e \int_{\Omega^e} (\mathbf{a} \cdot \nabla w) \tau [(\mathbf{a} \cdot \nabla) \Delta \rho - \nabla \cdot (\vartheta \nabla) \Delta \rho + \sigma \Delta \rho] d\Omega) \\ = \frac{1}{2} (w, (s^{n+1} + s^n)) + \frac{1}{2} (w, (h^{n+1} + h^n))_{\Gamma_N} + (a(w, \rho^n) - c(\mathbf{a}; w, \rho^n) - (w, \sigma \rho^n)) \\ + \sum_e \int_{\Omega^e} (\mathbf{a} \cdot \nabla w) \tau s d\Omega \end{aligned}$$

Where:

$$a(w, \rho) = \int_{\Omega} \nabla w \cdot (\vartheta \nabla \rho) d\Omega, c(\mathbf{a}; w, \rho) = \int_{\Omega} w(\mathbf{a} \cdot \nabla \rho) d\Omega, (w, \sigma \rho) = \int_{\Omega} w \sigma \rho d\Omega,$$

$$(w, s) = \int_{\Omega} w s d\Omega, (w, h)_{\Gamma_N} = \int_{\Gamma_N} w h d\Gamma.$$

2. R22 method

For R22 method the following scheme is used:

$$\frac{\Delta \rho}{\Delta t} - \mathbf{W} \Delta \rho_t = \mathbf{w} \rho_t^n$$

Where $\Delta \rho_t = \frac{\partial}{\partial t} \Delta \rho$, $\Delta \rho = \begin{Bmatrix} \rho^{n+\frac{1}{2}} - \rho^n \\ \rho^{n+1} - \rho^{n+\frac{1}{2}} \end{Bmatrix}$, $\mathbf{W} = \frac{1}{24} \begin{bmatrix} 7 & -1 \\ 13 & 5 \end{bmatrix}$, $\mathbf{w} = \frac{1}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$.

Adding SUPG space discretization, the scheme takes view:

$$\left(w, \frac{\Delta \rho}{\Delta t} \right) - (w, \mathbf{W} \Delta \rho_t) + \sum_e (\tau \mathcal{P}(w), \mathcal{R}(\rho))_{\Omega^e} = (w, \mathbf{w} \rho_t^n)$$

Where $\mathcal{R}(\rho) = \frac{\Delta \rho}{\Delta t} - \mathbf{W} \Delta \rho_t - \mathbf{w} \rho_t^n$ and $\mathcal{P}(w) = \mathbf{W}(\mathbf{a} \cdot \nabla) w$. Putting $\rho_t = -\mathbf{a} \cdot \nabla \rho + \nabla \cdot (\vartheta \nabla \rho) - \sigma \rho + s$ into the equation, receive:

$$\begin{aligned} & \left(w, \frac{\Delta \rho}{\Delta t} \right) + a(w, \mathbf{W} \Delta \rho) + c(\mathbf{a}; w, \mathbf{W} \Delta \rho) + (w, \sigma \mathbf{W} \Delta \rho) \\ & + \sum_e \int_{\Omega^e} \mathbf{W}(\mathbf{a} \cdot \nabla) w \tau \mathbf{W} [(\mathbf{a} \cdot \nabla) \Delta \rho - \nabla \cdot (\vartheta \nabla) \Delta \rho + \sigma \Delta \rho] d\Omega \\ & = (w, (\mathbf{w} s^n + \mathbf{W} \Delta s)) + (w, (\mathbf{w} h^n + \mathbf{W} \Delta h))_{\Gamma_N} + (a(w, \mathbf{w} \rho^n) - c(\mathbf{a}; w, \mathbf{w} \rho^n) - (w, \sigma \mathbf{w} \rho^n)) \\ & + \sum_e \int_{\Omega^e} \mathbf{W}(\mathbf{a} \cdot \nabla) w \tau s d\Omega \end{aligned}$$

5. - c) The number of elements is 20 in each direction, $a = 1, \vartheta = 0.001, \sigma = 0.001, s = 0, t^n = 0.5$. For initial time we receive the following figure:

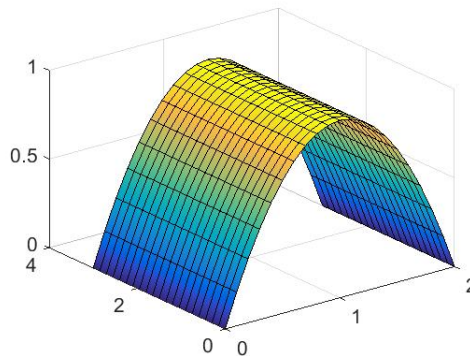


Fig 1: Initial time t=0

As it can be seen from the Fig 1, initial solution does not satisfy to boundary condition in $U\Gamma 4$. To fix it, we can use initial solution $(2-x)/2$.

The mesh with direction of velocity takes the following view:

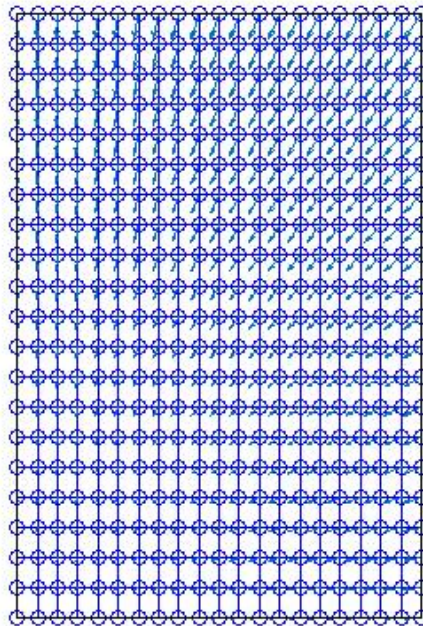
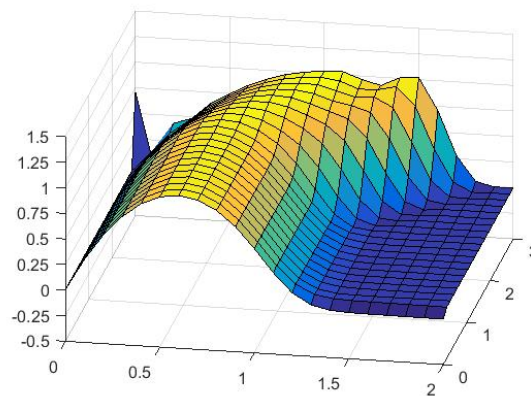
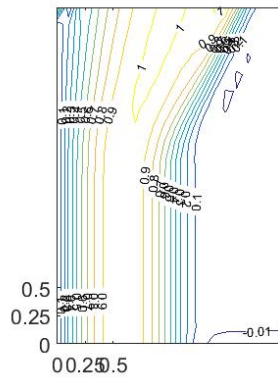


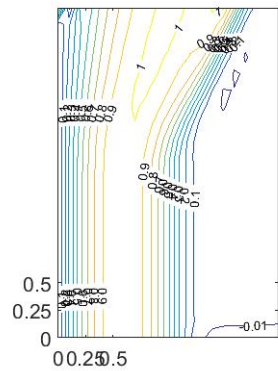
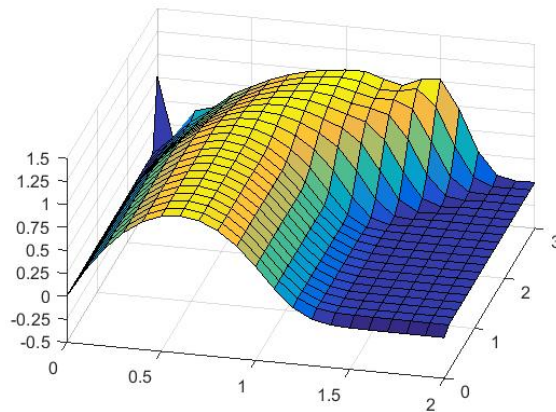
Fig 2: mesh

Solving the problem with Crank-Nikolson method for time discretization and SUPG method for spatial discretization:



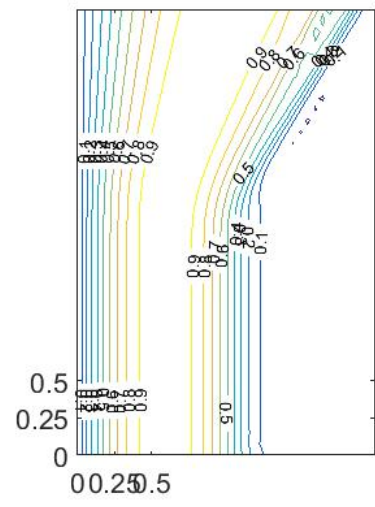
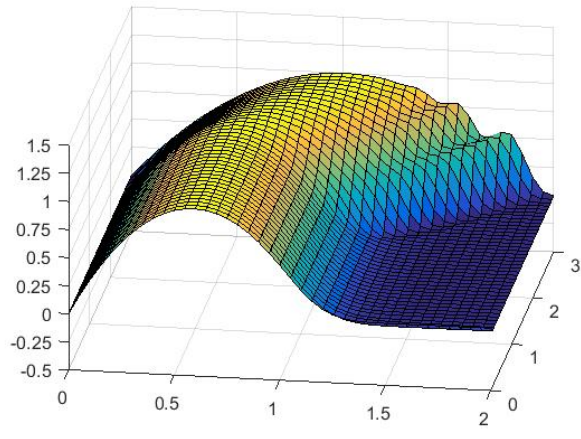


Solving the problem with R22 method for time discretization and SUPG method for spatial discretization:

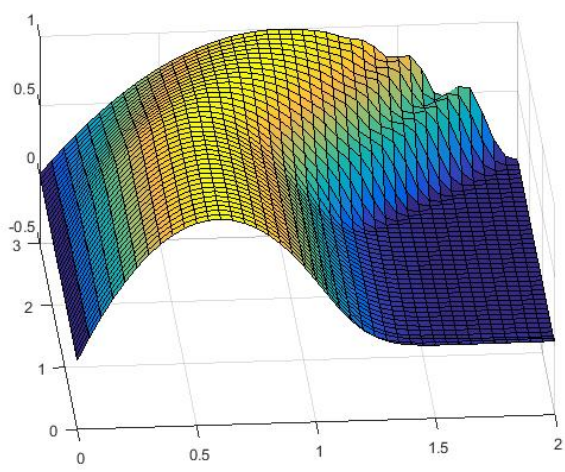


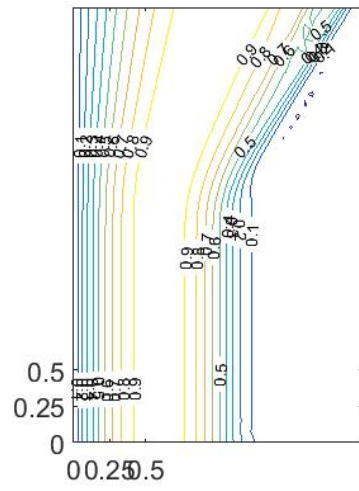
As can be seen from the figures above, the solution for both methods are similar. There also some fluctuations near the boundary can be observed.

d) Let us consider the same problem for quadratic elements. For CN+SUPG:



For R22+SUPG:





The both solutions are similar again. They are also similar to the solutions with linear elements, however there is no fluctuations on the corner.